MATHEMATICAL GEOGRAPHY

VOLUME II

SIMPLE ASTRONOMICAL AND TRIGONOMETRIC SURVEYING, AND THE MORE ADVANCED STUDY OF MAP PROJECTIONS

MATHEMATICAL GEOGRAPHY

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LONDON
SIR ISAAC PITMAN & SONS, LTD.
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PRINTED IN GREAT BRITAIN
AT THE PITMAN PRESS, BATH

PREFACE

This book is an extension of an earlier volume on this subject by the same authors. The aims and aspirations of the authors have already been set forth in the earlier book, and it is unnecessary to repeat them here. It is hoped, however, that the book will be found useful to those taking the special subject, "Cartography," in the Honours degrees of the University of London, as well as for other higher examinations in Mathematical Geography. It is also designed to give teachers who teach this subject in schools that higher knowledge which will enable them to present the more elementary teaching with greater confidence and in a more interesting manner.

A. H. J.
M. T. M. O.

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MATHEMATICAL GEOGRAPHY

CHAPTER I

SIMPLE ASTRONOMY

Object of Section.

In the first part of this book, we have explained the general principles on which measurements are made on the ground for the purpose of map drawing. We have said that for all extensive surveys a " triangulation " is necessary for a good result. By this method a number of isolated stations are fixed with great accuracy, and the detailed surveys are " adjusted " to fit properly between these on the paper.

Now when it comes to plotting the triangulation survey, we have seen that it is necessary to choose some particular projection for the map. No map *can* be perfect in all respects, and it is not always easy to choose the best projection, even for a particular purpose.

When, however, the choice has been made, we are able to draw the lines of latitude and longitude on the map, according to the principles already outlined, and elaborated in this volume. Having drawn these, it will be clear that we can plot a triangulation station on the map, in its proper place, *if we know its latitude and longitude*. But we cannot even start the map until we *do* know, at any rate, the latitudes and longitudes of *some* of the triangulation stations.

Now these have to be determined by astronomical observations, and the object of this section is to explain the principles on which this is done, and the details of some very simple methods. The instrument chiefly referred to will be the theodolite. A diagram of a high-class theodolite by Messrs. Cooke of York is given on page 52, with a description which will enable the student to learn the names of the parts.

Cheaper instruments can be obtained, of course; but in the opinion of the writer it is wiser to buy even a very old second-hand instrument of good original construction than a cheap new one.

Definitions.

It is important that the student should be quite clear as to the meanings of the different terms to start with. Assuming, for the moment, that the earth is a true sphere, and that the station with which we are concerned is a point A on the surface, the *vertical* is defined as the line AO joining our station to the earth's centre. Thus the station A is, for us, the *top* of the earth, and AO is vertical.

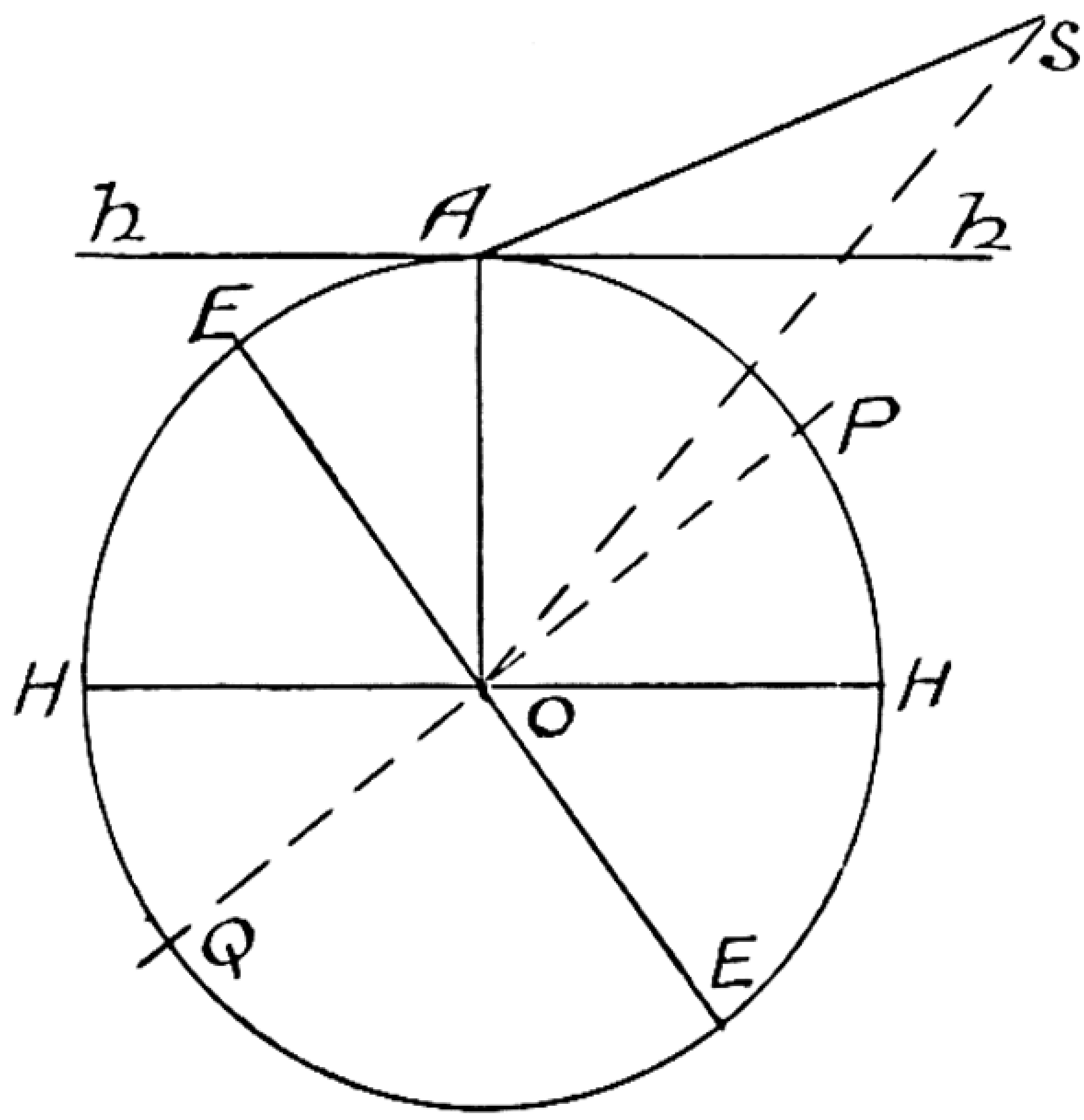


FIG. 1

The *horizon* is defined as a plane hAh passing through A at right angles to the vertical.

Altitude.

The *altitude* of any point S from A, in ordinary terrestrial surveying, means the angular height of the point S above the horizon. Thus the angle hAS is the altitude of S, assuming that the latter is in the plane of the paper.

It can be measured, roughly, by clinometer; or, more accurately, by sextant, theodolite, etc.

Astronomically, all altitudes are supposed to be measured at the centre of the earth, instead of at A. That is, if S be a heavenly body, its altitude is defined as the angle HOS (Fig. 1) instead of hAS, where HH is parallel to hh.

If S is a "star," its distance from the earth is so great that the earth's radius can be neglected by comparison, and the angle HOS is the same as hAS. But if it is a nearer body, such as the sun or moon, a correction must be applied to the

measured altitude for a good result. This is called the *parallax* correction. It is always to be *added*, because it will be clear that the body would appear *higher* if seen from O than from A. The amount varies with the altitude and with the distance of the body observed, but can be found from tables.

Latitude.

The line OA being vertical, it follows that the earth's polar axis will *not* be vertical, but will be inclined to the horizon at an angle depending on the position of the station A. In Fig. 1, P and Q are supposed to be the poles, so that PQ is the polar axis.

The equatorial plane passes through the centre O, at right angles to PQ, and meets the surface in the line EE. This line, like the horizon HH, is really a circle passing round the earth, but is shown here as a straight line because it is supposed to be viewed edgeways.

The angular distance of the station north or south of the equator is called the *latitude* of the station. Thus the angle EOA represents the latitude of A.

Latitude is always measured at right angles to the equator, and therefore in a plane passing through the pole.

If a plane be drawn through A parallel to the equator, it meets the sphere in a small circle Aa (Fig. 2), which is called the *parallel of latitude* through A. If a be any point on this parallel, the angle eOa is the latitude of a, and is the same as EOA. This is evident because if we simply turn the quadrant PAE round PQ as axis, A moves to a, and E moves to e, without any change in the angles. Hence all points on the same parallel have the same latitude.

The student must begin by getting clearly hold of the difference between altitude and latitude. The matter is a little confused because geography books often show the polar axis vertical. In this case, the equator would coincide with the astronomical horizon HH, and latitude and altitude would then be measured from the same plane.

But the polar axis is *not* vertical unless one is at the pole; hence the equator is *not* horizontal, and latitude and altitude are quite different things. All points on the same parallel have the same *latitude*, but the *altitude* varies from point to point.

Meridian.

The plane containing the polar axis and any given point is called the *meridian* of the point. Thus *PAE* is the meridian of *A* and *Pae* is the meridian of *a* (Fig. 2).

The meridian of the supposed station *A* also contains the vertical line *AO*, and therefore is itself a vertical plane. It is represented in the diagrams by the plane of the paper.

The direction of the meridian is north and south at *A*.

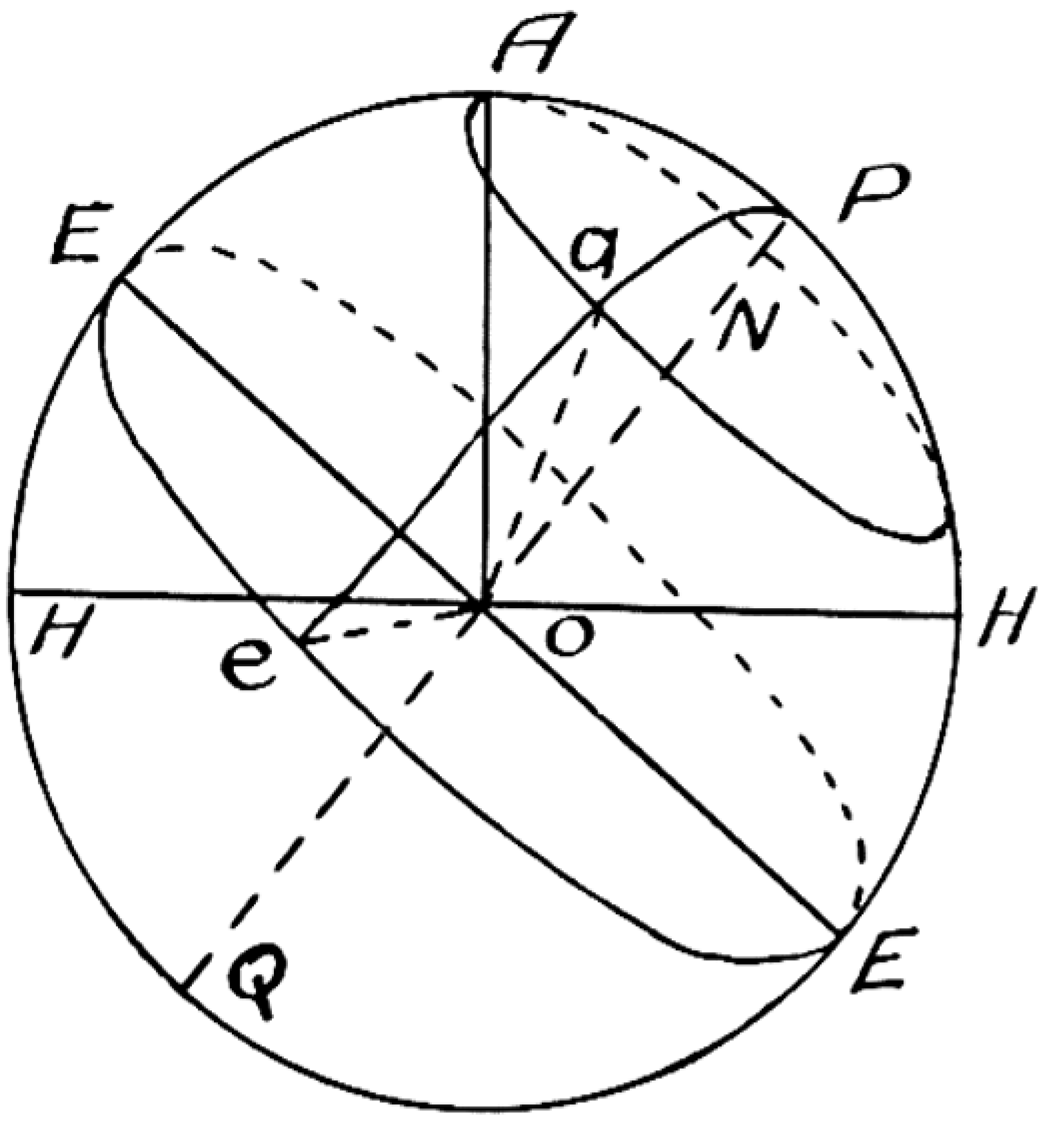


FIG. 2

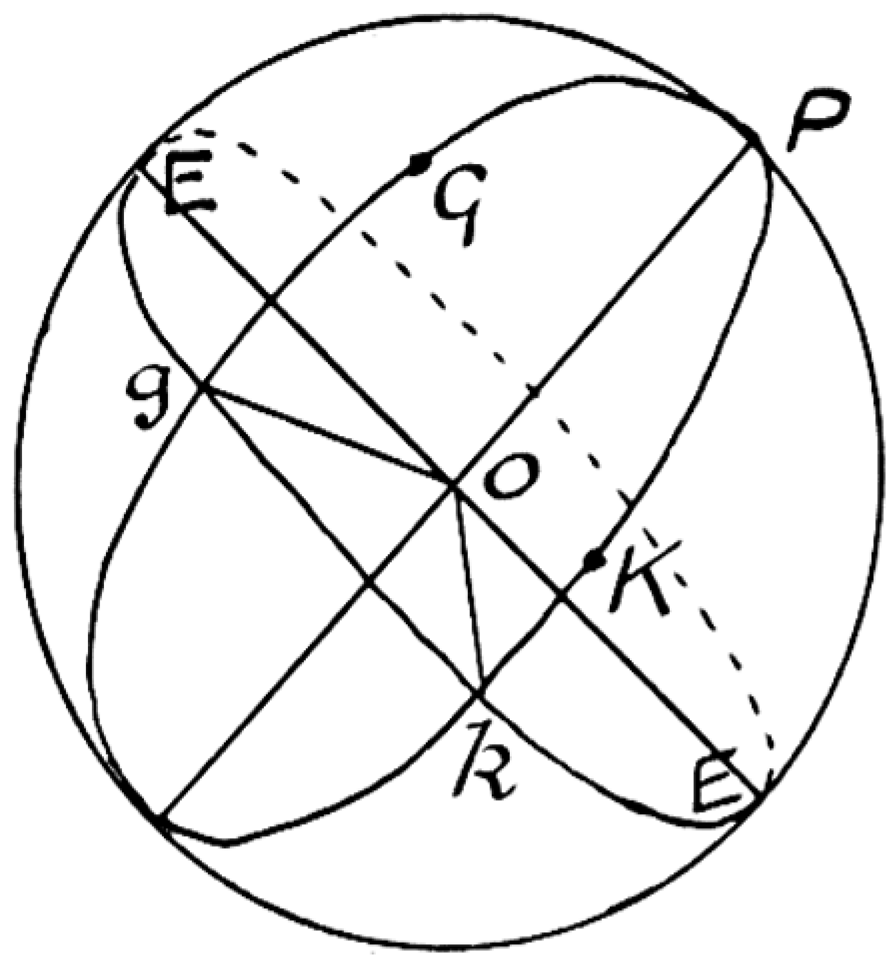


FIG. 3

Hence the meridian plane at any station gives the direction of the north and south line there.

Longitude.

For measuring longitude, one meridian is chosen as standard (the meridian of Greenwich), and the longitude of any other point means the angular distance between the meridian of the point and the standard meridian. This angle is measured at the pole. Thus if *G* (Fig. 3) be supposed to represent Greenwich, and *K* any other point, the angle *GPK* is the longitude of *K*. It is measured east or west of the standard, from 0 to 180°. In the case shown, if *P* be the north pole, an observer at *G*, looking north, will have *K* on his right hand, so *K* is east of Greenwich. If the meridians *PG*, *PK* be produced to meet the equator at *g* and *k*, and these points

be joined to O, the angle gOk is the same as the angle GPK on the sphere ; hence longitudes can be equally well measured round the equator.

Celestial Sphere.

Everybody nowadays knows, of course, that the different heavenly bodies are dispersed in space ; that they all *appear* to move round the earth in the same direction, *because* the earth is actually rotating round its polar axis in the *opposite* direction ; and that they also appear to move more or less with respect to one another in consequence of the movement of the earth with and round the sun, and of their own individual movements.

The beginner, however, finds it difficult to reason on these lines. Hence it is usual to suppose that the earth is fixed (and, with it, the various lines and planes we have been considering) and that the heavenly bodies are set on the surface of an outer surrounding sphere, which rotates in the opposite direction to that in which the earth actually turns, but round the same polar axis.

For most practical purposes this comes to the same thing, and it is easier to imagine. This outer sphere is called the stellar or celestial sphere, and is concentric with the earth. The various lines and planes we have been discussing can be produced to meet it. This is shown in Fig. 4, where the inner circle represents the earth. The terrestrial polar axis and equator are produced to give the corresponding features on the celestial sphere. The horizon HH is also produced to form the celestial horizon. It remains fixed with the earth, and as the stellar sphere rotates, the different bodies cross it as they rise and set.

Similarly the vertical OA is produced to fix the "zenith," Z. The vertical OA remains fixed with the station A, so that Z is also a fixed point, and does *not* move round with the stellar sphere. The same is true of the meridian of A. This plane produced passes through Z and P on the outer sphere, and it remains fixed with the station A, and is represented by the plane of the paper.

We have stated that the earth's radius is negligible by comparison with the distance of the stars, and that for nearer bodies all observations are corrected to reduce them to the

earth's centre. Hence it is usual to omit the earth altogether from the diagram, and to show the stellar sphere only. The student must carefully remember, however, that the vertical and horizontal directions, and that of the meridian, or north-and-south line, are fixed by the station on the earth, even though it is not shown. We have seen that EOA is the latitude of A. It is clear that this is the same angle as EOZ on the celestial sphere. Hence when we draw the celestial sphere only, the angle EOZ represents the latitude of the station.

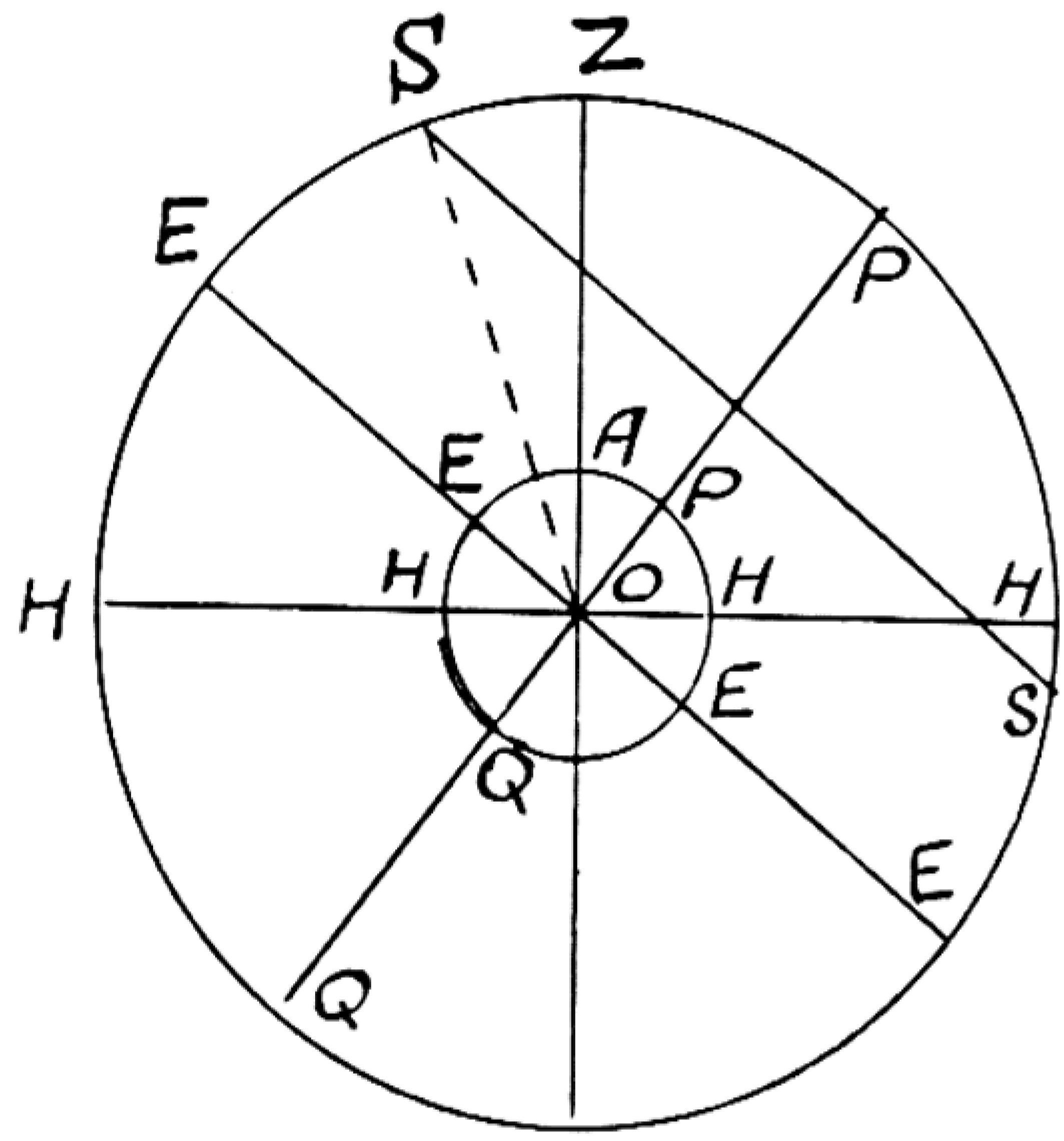


FIG. 4

As the stellar sphere rotates, each star describes a small circle, such as Ss, parallel to the equator, and crosses the meridian *twice*, at S and s. It is then said to *transit*. The point S is called *upper culmination*, because the star is there at its *greatest* height above the horizon, and at s it is said to be at lower culmination.

Declination.

The angular distance of a star, north or south of the equator, is called the *declination* of the star. It corresponds with *latitude* for a terrestrial point. Thus EOS (Fig. 4) is the declination of the star S.

This declination remains nearly constant for the distant stars, and its value is given in the *Nautical Almanac* for each of the principal stars at intervals of 10 days throughout the year.

In the case of the sun, moon, and planets, the respective motions cause a more rapid change in declination, because the bodies are so much nearer to us. Hence the declination is given at more frequent intervals; for example, daily for the sun, hourly for the moon.

In consequence of the movement of the earth's polar axis,

stars near the poles also show more rapid variations, and their positions are given daily.

For practical purposes it is necessary that the surveyor should be able to identify a star when he sees it in the heavens. This can be done with the aid of star charts, and we shall suppose that the student is able to do this.

DETERMINATION OF LATITUDE

Latitude by Meridian Altitude.

Let us suppose, then, that at a station in the northern hemisphere some known star has been chosen, and that its altitude is observed by the theodolite just at the moment when it is crossing the meridian, that is when it is due south from the station. If S_1 (Fig. 5) be the star, its altitude is HOS_1, and this is called a *meridian altitude*, as the star is just crossing the meridian. Hence HOS_1 is known.

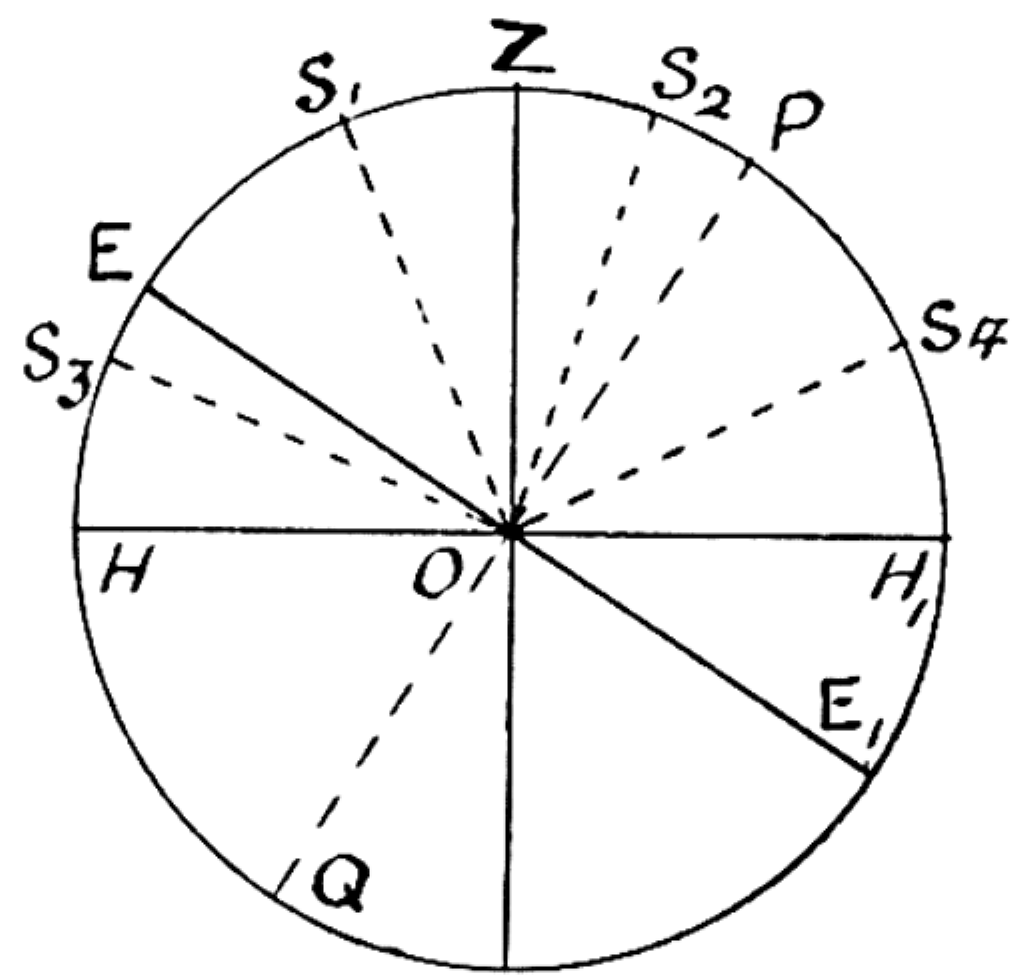


FIG. 5

Now the star's declination is indicated by the angle EOS_1, and is obtainable from the *Nautical Almanac*. And we have seen (page 6) that the required latitude of the station is represented by the angle EOZ.

Hence the problem is, given the angles HOS_1 (= altitude), and EOS_1 (= declination) required to find EOZ (= latitude).

Now $HOZ = 90°$; hence $S_1OZ = 90° - HOS_1 = 90° -$ altitude. This angle S_1OZ is sometimes called the *zenith distance* and denoted by z.

Now $EOZ = EOS_1 + S_1OZ$, obviously.

Hence we have

latitude = declination of star + zenith distance,

or $$\lambda = \delta + z \qquad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where λ = latitude of station (required)
δ = declination of star observed (from *N.A.*)
$z = 90° -$ altitude (observed with theodolite).

Thus the formula for a latitude determination is very simple, and should be learned.

Unfortunately, however, the star may not be in a position like S_1. If one is chosen near the pole, it may transit between Z and P, as shown at S_2. In this case, the star is *north* of the observer, and the altitude is H_1OS_2, and the declination is EOS_2.

As before, $z = ZOS_2 = 90° -$ altitude.

But now, latitude $= EOZ = EOS_2 - ZOS_2$.

$$\therefore \lambda = \delta - z \qquad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

To the mathematician this change presents no difficulty. The zenith distance z is measured in the opposite direction to what it was before, so it must be reckoned *minus*.

Next we may have a star in such a position as S_3.

Here HOS_3 is the altitude, and $S_3OZ = z = 90° -$ altitude. EOS_3 is the declination, $= \delta$.

And, clearly $EOZ = S_3OZ - S_3OE$,

or $$\lambda = z - \delta \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

It may appear, at first sight, difficult for the observer to distinguish between these cases, as the equator is not marked in the sky. But as a matter of fact, if we are in the northern hemisphere a star at S_1 will have *north* declination, and one at S_3 will have *south* declination. And as the *Nautical Almanac* tells us which way the declination is, there is no difficulty in distinguishing.

Note that here the *declination* has changed sign as compared with S_1, so that δ becomes *minus*.

Finally we may have a star at S_4, where it is below the pole, at lower culmination, and north of the observer.

In this case, the altitude is H_1OS_4, and $z = ZOS_4$; the declination is E_1OS_4.

Now clearly $EOZ = 180° - (ZOS_4 + E_1OS_4)$,

or $$\lambda = 180° - \delta - z \qquad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Here δ is measured from E_1 instead of E, so that, by comparison with case No. 2, δ becomes $180° - \delta$.

These four formulae cover all possible cases.

TABLE OF APPROXIMATE RIGHT ASCENSIONS AND DECLINATIONS OF STARS

(From the *Nautical Almanac* for 1926)

Star	Right Ascension	Declination	Star	Right Ascension	Declination
	h. m.	° ′		h. m.	° ′
α Androm.	0 5	28 41 N.	Regulus	10 4	12 20 N.
β Cassiopeiae	0 5	58 45 N.	η Argus	10 42	59 18 S.
α Phoenicis	0 23	42 42 S.	β Ursae Maj.	10 57	56 47 N.
α Cassiopeiae	0 36	56 8 N.	Dubhe	10 59	62 9 N.
β Ceti	0 40	18 24 S.	Denebola	11 45	14 59 N.
γ Cass.	0 52	60 19 N.	γ Ursae Maj.	11 50	54 6 N.
β Androm.	1 6	35 14 N.	α Crucis	12 22	62 41 S.
α Polaris	1 35	88 54 N.	γ ,,	12 27	56 42 S.
Achernar	1 35	57 37 S.	γ Centauri	12 37	48 33 S.
γ Androm.	1 59	41 59 N.	β Crucis	12 43	59 17 S.
α Arietis	2 3	23 7 N.	ε Ursae Maj.	12 51	56 22 N.
Algol	3 3	40 40 N.	Spica	13 21	10 47 S.
α Persei	3 19	49 36 N.	η Ursae Maj.	13 45	49 41 N.
Aldebaran	4 32	16 22 N.	β Centauri	13 59	60 1 S.
Rigel	5 11	8 17 S.	θ ,,	14 2	36 0 S.
Capella	5 11	45 55 N.	Arcturus	14 12	19 34 N.
Bellatrix	5 21	6 17 N.	α Centauri	14 35	60 32 S.
β Tauri	5 22	28 33 N.	β Ursae Min.	14 51	74 27 N.
ε Orionis	5 32	1 15 S.	α Coronae Bor.	15 32	26 58 N.
ζ ,,	5 37	1 59 S.	δ Scorpii	15 56	22 25 S.
κ ,,	5 44	9 42 S.	Antares	16 25	26 16 S.
Betelguese	5 51	7 24 N.	β Herculis	16 27	21 39 N.
β Aurigae	5 54	44 57 N.	α Triang. Aust.	16 41	68 54 S.
β Canis Maj.	6 19	17 55 S.	α Herculis	17 11	14 28 N.
Canopus	6 22	52 39 S.	λ Scorpii	17 29	37 3 S.
γ Geminorum	6 33	16 28 N.	α Ophiuchi	17 31	12 37 N.
Sirius	6 42	16 37 S.	θ Scorpii	17 32	42 57 S.
ε Canis Maj.	6 56	28 52 S.	κ ,,	17 37	39 0 S.
δ ,, ,,	7 5	26 16 S.	γ Draconis	17 55	51 30 N.
η ,, ,,	7 21	29 9 S.	ε Sagitarii	18 19	34 25 S.
Castor	7 30	32 3 N.	Vega	18 34	38 43 N.
Procyon	7 35	5 25 N.	Altair	19 47	8 40 N.
Pollux	7 41	28 12 N.	γ Cygni	20 20	40 1 N.
ζ Argus	8 1	39 48 S.	α Pavonis	20 20	56 58 S.
γ ,,	8 7	47 7 S.	Deneb	20 39	45 1 N.
ε ,,	8 21	59 16 S.	ε Pegasi	21 41	9 32 N.
δ ,,	8 43	54 26 S.	α Grius	22 4	47 19 S.
λ ,,	9 5	43 8 S.	β ,,	22 38	47 16 S.
β ,,	9 12	69 25 S.	α Piscis Aust.	22 54	30 1 S.
i ,,	9 15	58 58 S.	β Pegasi	23 0	27 41 N.
α Hydrae	9 24	8 20 S.	α ,,	23 1	14 48 N.

A table showing the approximate declinations of a few bright stars for 1926 is given on page 9. With the aid of this table, it is proposed now to work a few examples, after which we shall say something about the actual observation.

The student who has leisure will find it interesting to plot these stars on a sinusoidal or Mollweide projection. Every fifteen degrees on the equator must be counted as one hour, and the numbering for right ascension must run from right to left. (See page 28.)

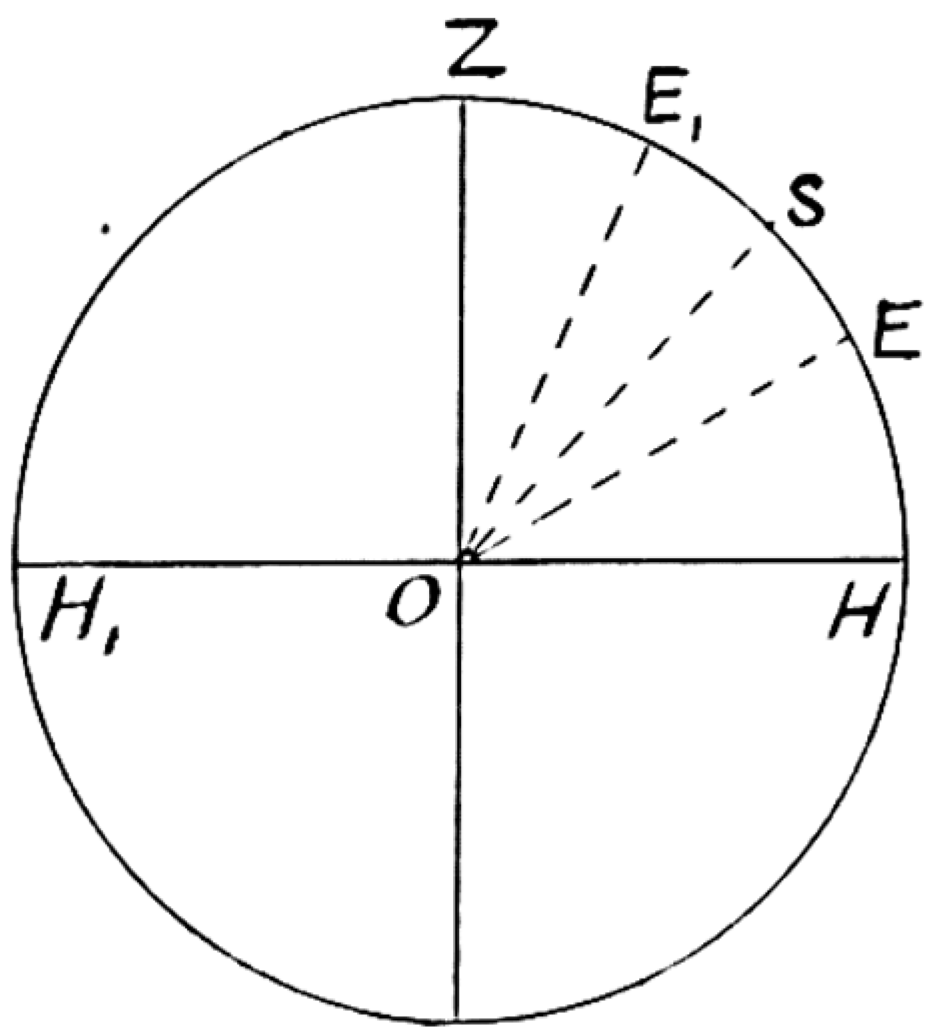


FIG. 6

Examples.

(1) At a station in the northern hemisphere, the meridian altitude of Aldebaran is observed as 37° 42′. Find the latitude of the station.

In solving such exercises, always begin by drawing the vertical OZ (Fig. 6) and the horizon H_1H. These are always the same. Next place the star on the figure, according to its observed altitude.

This may be put *either* on the left or right-hand side, as it is merely a question of which way we suppose we are looking at the figure. In this case, let us put it on the right so that $HOS = 37° \ 42'$. Next the equator must be placed.

Now the angular distance from the equator to the star is known, being the *declination*. From the table on page 9 we see that the declination of Aldebaran is 16° 22′ N. approximately.

Hence the angle SOE between the star and the equator is 16° 22′. And the equator must be *below* the star, as shown at EO, because the station is in the *northern* hemisphere, and the declination is also *north*. If we were to place the equator at OE_1, above the star, then Z (which represents the station) and the star would be on opposite sides of the equator, and hence could not both be north.

Therefore OE is the equator, and we have $HOS = 37° \ 42'$; $EOS = 16° \ 22'$. From these we require to find EOZ, which represents the latitude.

$$\begin{aligned} \text{Clearly } SOZ &= 90^\circ - 37^\circ\ 42' = 52^\circ\ 18' \\ EOS &= 16^\circ\ 22' \\ \therefore\ EOZ = SOZ + EOS &= 68^\circ\ 40' \text{ N.} \end{aligned}$$

In this case there is no ambiguity. But in some cases ambiguity may arise unless we know whether the star transits north or south of zenith, or whether it is at upper or lower culmination. Hence it is well to state these facts at the time.

(2) At a certain station the meridian altitude of β Ursae Minoris is observed as $56^\circ\ 24'$. Find all possible values of the latitude.

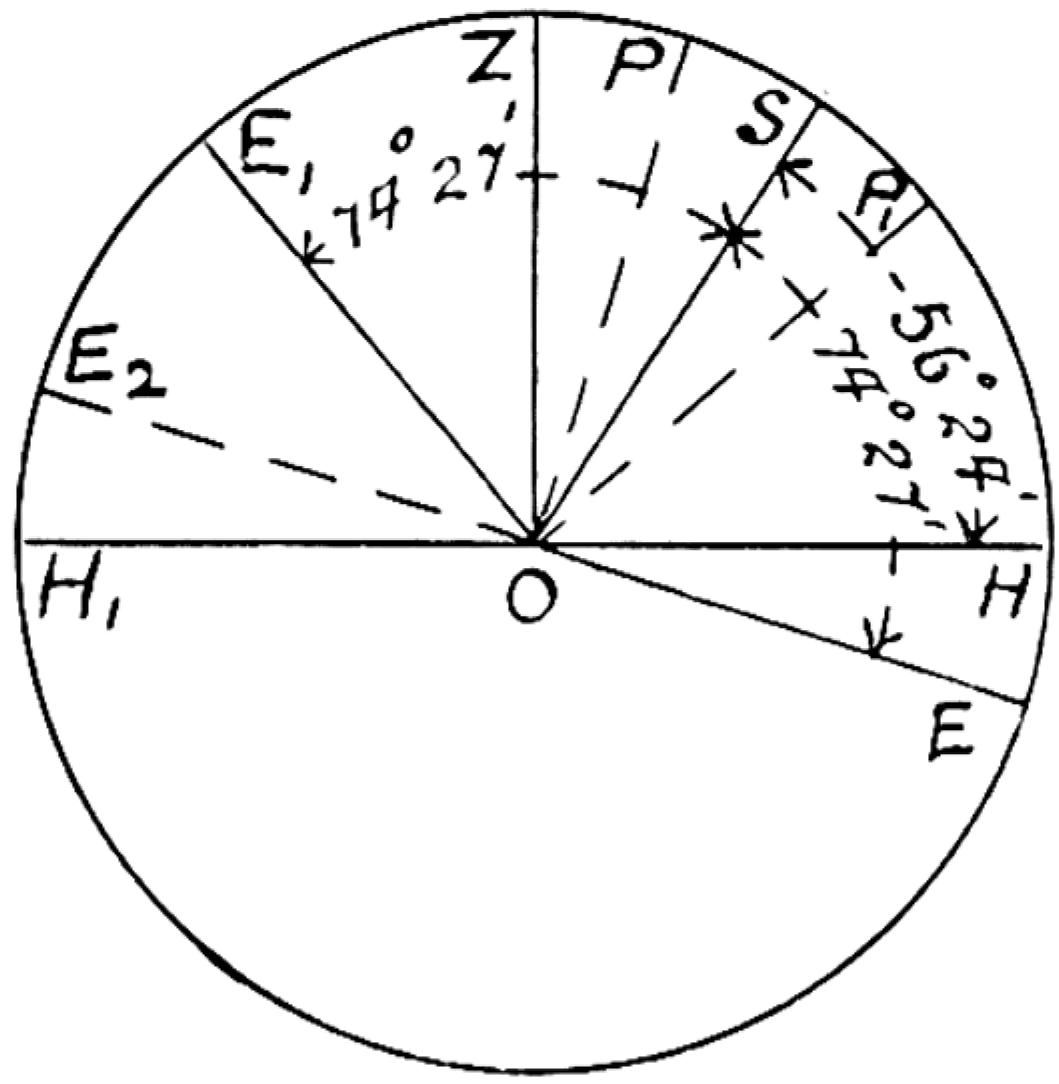


FIG. 7

Here HOS (Fig. 7) = $56^\circ\ 24'$. As nothing is said about where the station is, or which way we are looking, we may place the equator *either* at E or at E_1 so that $EOS = E_1OS$ = $74^\circ\ 27'$, the latter being the star's (north) declination from the table.

In the first case, produce EO as shown at OE_2. Then E_2OZ is the latitude, as this is always measured the *nearest way* from the equator, and therefore never exceeds 90°

$$\begin{aligned} \text{Now} \quad E_2OS &= 180^\circ - EOS = 105^\circ\ 33' \\ ZOS &= 90^\circ - 56^\circ\ 24' = 33^\circ\ 36' \\ E_2OZ &= 71^\circ\ 57' \end{aligned}$$

The latitude is *north*, as Z is on the same side of the equator as the star. The pole would be at P, so that $EOP = 90^\circ$.

Hence this is a case of lower culmination, the star being below the pole.

In the second case, OE_1 is the equator, and E_1OZ the latitude. Evidently latitude

$$= E_1OS - ZOS$$
$$= 74^\circ\ 27' - 33^\circ\ 36' = \underline{\underline{40^\circ\ 51'}} \text{ north.}$$

The pole would be at P_1, so that $E_1OP_1 = 90^\circ$.

(3) In example No. 1, find the latitude if the station is in the southern hemisphere.

Here E_1OS (Fig. 6) must be the equator, as the star is known to be *north*, and the station Z is *south*.

Latitude $= E_1OZ = 52^\circ\ 18' - 16^\circ\ 22' = 35^\circ\ 56'$ south.

There is usually no difficulty in stating whether we are looking north or south. Most people know the pole star, and we can tell whether we are facing it or have it behind us. Failing this, a compass will give the information. Similarly, it is easy to say whether the star is below the pole (lower culmination) or above it.

Taking the Observation.

The conditions to be satisfied by the observations may be placed under two headings. In the first place, the star must be observed at the moment when it is crossing the meridian.

We shall show later that the *time* at which it crosses is known, so that, if we know local time accurately, we can observe it at that time.

We have also stated that a star is either due north or due south of the observer when it transits ; hence if we know the direction of a line due north or south of the station, we can observe the star when it is in that direction.

As we can always lay out this line approximately by any of the methods described in the first volume of this book, and as the local time is always fairly well known, it is easy to make the observation *nearly* in the right place. A slight error is of little importance in school work, as the star is moving horizontally when it crosses, and the altitude does not vary rapidly then.

It may be well, however, to observe the star a few minutes *before* the proper time, and bring it on to the horizontal hair of the theodolite. It should then be rising slowly, but as the

theodolite (with the usual eyepiece) reverses things, it should *appear* to be *falling*. We work the tangent screw of the theodolite to bring it back on the wire, and keep on doing this as long as it appears to fall. When crossing the meridian it is at its highest point and moves along the horizontal wire; as soon as it begins to rise *above* the wire, we know that it has turned and is falling, and we follow it no longer. The altitude is then read on the vertical circle.

The light from the star is not sufficient to show up the cross hairs. Some theodolites are fitted with special reflectors for this purpose, but it is always possible to see them by the aid of a carefully held lamp shining into the object glass at a slight angle.

In the second place, the altitude is required free from all errors. Now the chief sources of error are instrumental errors, atmospheric refraction, and personal errors.

Instrumental errors arise from the fact that the instrument is not properly levelled and from errors of adjustment, whereby the vertical circle does not read zero when the line of sight is horizontal. This error is called an *index error*; its value and sign may be found by reading the angle of elevation to any convenient *fixed* point, both *face-left* and *face-right* (see Vol. I, p. 79), taking care that the instrument is accurately levelled for both readings. The exact method of levelling varies with the make of theodolite, and must be studied on the instrument itself. If there is a spirit level attached to the vernier arm of the vertical circle, the levelling must be done by that level, and it must be made central for each reading.

Half the difference between the angles of elevation as read above will give the index error.

For example, if we read a fixed point as follows—

Face-left, angle of elevation	=	9° 27′
Face-right, ,, ,,	=	9° 43′
Difference	=	16′
Index error	=	8′

This is to be *added* to all angles of elevation observed with the instrument *face-left*, as the face-left reading was too small. Several readings should be taken.

Atmospheric refraction is due to the fact that rays of light from the star to the observer do not travel in straight lines, through the air, but in curved ones, as the air varies in density from point to point.

This always makes the star appear *higher* than it really is. The actual amount varies with the star's altitude and with the exact state of the atmosphere.

Hence it is always a little uncertain. A table showing approximate values at different altitudes is given on page 41, and fuller tables will be found in books of mathematical tables.

This correction is always to be *subtracted* from the observed altitude.

Personal errors arise through bad setting of the hair on the star, and so on, and diminish with practice.

For a good result, of course, the correct declination must be obtained for the date, and the approximate figures in this book must not be employed.

The best and most accurate method of finding latitudes in practice are merely elaborations of this method. Stars are chosen as near the zenith as possible, so as to diminish refraction, and they are observed in pairs (on opposite sides of the zenith, but about equally distant from it, and taking care that both are observed with the same *face* of the theodolite. The effect of this is that the zenith distance is positive for one star and negative for the other. Thus, on the mean of the two, any error due *either* to faulty adjustment of the theodolite *or* refraction is likely to cancel out, while the effect of personal and accidental errors is reduced by taking a number of stars.

We have seen that for a star at S_1 (Fig. 5, page 7)—

$$\lambda = \delta_1 + z_1$$

and for a star at S_2, $\quad \lambda = \delta_2 - z_2$.

Hence on the mean, $\lambda = \frac{1}{2}(\delta_1 + \delta_2) + \frac{1}{2}(z_1 - z_2)$.

In Talcott's method for latitudes, a special instrument, called a "zenith sector," is used, and we do not measure the actual altitudes at all, but only the *difference*, $z_1 - z_2$, by means of a micrometer.

But the *principle* of the method is simply that of a pair of meridian altitudes.

Sun Observations.

Where the sun is the body observed, the altitude must be read at *noon*. This does not necessarily mean at 12 o'clock. The time must be corrected for longitude and for equation of time, as described in Vol. I, pages 27 and 28. If the time is at all doubtful, it can be observed a little before noon, and followed in the same way as described for a star.

There are several other points of difference, however. In the first place, the sun's centre is not marked in any way, so that we must take the observation on the *edge* of the disc, either the lower edge or the upper. If we *appear* (with the ordinary theodolite eyepiece) to be observing the lower edge, as in Fig. 8, then we are *really* observing the *top*, as the theodolite reverses things.

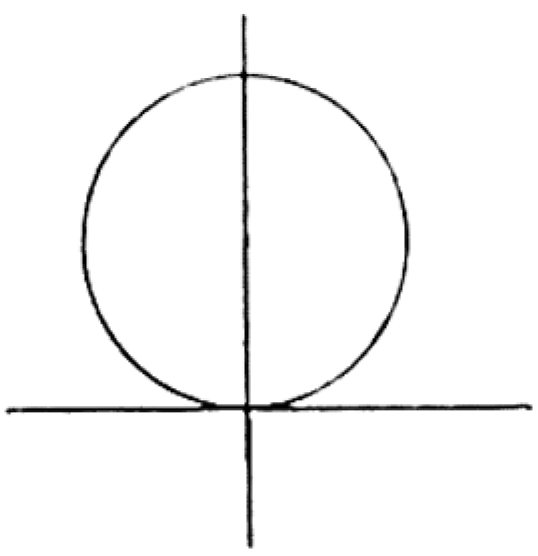


FIG. 8

A correction must be applied accordingly for the angular value of the sun's radius or "semi-diameter," which is given daily in the *Nautical Almanac*, and varies from 16′ 17½″ on 3rd Jan. to 15′ 45⅓″ on 5th July.

In the case shown this would have to be *subtracted* from the observed altitude to find the altitude of the centre. A dark glass must be used for sun observations, attached to the eyepiece. With this, the cross hairs cannot be seen unless they actually cross the sun's disc. Hence, if the sun is being *followed* to find the highest altitude, he should always be observed *above* the hair, as shown in Fig. 8. We are then really observing the top. As long as the sun is rising, he will appear to fall, and the horizontal hair will be seen crossing the disc, and must be brought back tangent to it. When he begins to fall, he will move away from the hair, and the latter will no longer be visible. We do *not* follow him in that case.

The horizontal circle should be turned to keep the vertical hair roughly through the sun's centre.

Moreover, the object glass should be partly covered when taking sun observations, as otherwise we may burn the spider's webs in the theodolite. The theodolite may be fitted with a cap for this purpose, but if not it can be covered with a piece of paper with a hole in it. The index error should be found

with the object glass covered in the same way, as the lens may not be perfect.

Then a correction must be made for parallax (p. 3), as the sun is comparatively near the earth. The formula for this correction is about $9 \times \cos$ altitude, in seconds, but as its amount is always less than 9 seconds, it can be neglected in rough work.

The exact value is given for different altitudes in *Chambers'* (and other) *Mathematical Tables*.

Refraction and parallax corrections should be applied before the semi-diameter correction, but index-error correction (if any) should be applied first of all.

Lastly, when observing the sun the declination varies more rapidly than for a star. Hence for a good result we must know the approximate Greenwich time of the observation, as well as the date, in order that the proper declination may be found. The *Nautical Almanac* gives the declination at Greenwich noon, and the rate of increase or decrease per hour.

Worked Example.

As an illustration, suppose that on 11th July, 1928, the altitude of the sun's *upper* edge when crossing the meridian was observed as 59° 12′ 30″, face-left, the index error being 8 minutes to be added. Find the latitude, given that the G.M.T. (p. 18) was 12 hr. 20 min. p.m. The corrections are as follows—

Observed altitude =	59° 12′ 30″	
Index error +	8′	
	59° 20′ 30″	
Refraction (*Chambers' Tables*)	34″	(always minus)
	59° 19′ 56″	
Parallax (ditto)	4″	(always plus)
	59° 20′ 0″	
Semi-diameter (*Naut. Alm.*)	15′ 45″	
Corrected altitude =	59° 4′ 15″	
$z = 90° -$ altitude =	30° 55′ 45″	

We have now to find the declination.

The *Nautical Almanac* tells us that at Gr. mean noon on that day the declination was $22° 6' 56''$ N., decreasing $20''$ per hour.

Hence at 12 hr. 20 min. it would be $7''$ less;

hence $\delta = 22° 6' 49''$;

and latitude $= z + \delta = 53° 2' 34''$ N.

Time and Longitude.

The period between two successive appearances of the sun, or of any given star, on the meridian of a station is called a *day*, and is divided into 24 hours. Now the body in question appears, in that period, to have passed right round the earth, or to have described 360 degrees of revolution, relatively to the station. This is equivalent to a rate of 15 degrees of longitude per hour of time.

Hence to find the longitude of any station, all we have to do is to find out, at any given instant, what the time is at Greenwich, and at the same instant the local time at the station. The difference in time is reduced to longitude as above; and as the bodies appear to move from east to west, the time is always *ahead* at the more easterly station. Hence we have the old sailor's rhyme:

Longitude east, Greenwich least.
Longitude west, Greenwich best.

The student must therefore make a clear mental distinction between Greenwich time and local time.

For example, if we can tell Greenwich time by a chronometer at the moment when it is noon at a station, we can find the longitude of the station.

Unfortunately, however, there are several different kinds of "noon," and of "time," and the matter is somewhat complicated in consequence.

Mean and Apparent Time.

For instance, at the moment when the sun's centre is just crossing the meridian of any station, it is said to be *apparent noon* at that station. In other words, the *local apparent time* is 12 noon. Even if the station were Greenwich, a properly regulated clock there would not, in general, indicate 12 o'clock then. At 12 o'clock exactly, it would be said to be *mean noon*.

The fact is that the period between two successive transits of the sun, across the same meridian, is not always the same. The sun moves faster sometimes than at other times, so that a solar day varies in length.

Now it would be a hopeless business to try to change the rate of a clock from day to day, so that it might always keep pace with the sun. Hence the *mean* length of a solar day, throughout the year, is calculated, and solar clocks are regulated to register 24 hours in this period. If such a clock is set to agree with the true sun at a certain station (say Greenwich) and at a certain moment, which is agreed upon beforehand for all places, then the time by such a clock is called *mean time* at the station.

The time by the true sun is called *apparent time*.

At certain times of the year the clock will be gaining on the sun, and at other times losing.

The time indicated by a sundial is *apparent time*, and may be either ahead of the clock or behind it.

The difference between mean time and apparent time is called the *equation of time*. Its value is given in the *Nautical Almanac*, for every day in the year, at the moment of Greenwich noon, accompanied by a statement as to whether it is to be added to, or subtracted from mean time.

Its rate of increase or decrease per hour is also given, so that it can be found at any moment, if the Greenwich time is known.

It is zero four times a year, in April, June, August, and December. The maximum values are $14\frac{1}{2}$ minutes in February (clock ahead of sun) and about $16\frac{1}{2}$ minutes in November (sun ahead of clock).

It will be evident that in finding longitude it will *not* do to use, say, the Greenwich *mean* time, along with the *apparent* time at the station. *Both* must be reduced to mean time before taking the difference.

The abbreviations G.M.T. and G.A.T. are used for Greenwich mean time and Greenwich apparent time. Similarly L.M.T. and L.A.T. stand for local time. The letters G.M.N., G.A.N., L.M.N., and L.A.N., stand similarly for Greenwich or local mean or apparent *noon*. Thus, L.A.N. means local apparent noon, which is the instant at which the sun's centre crosses the meridian of the station at upper culmination.

Example.

The sun was due south at a certain station in England on 16th March, 1926, and a Greenwich mean time chronometer at that moment read 12 hr. 17 min. 26 sec. p.m. Find the longitude, given that the equation of time at G.M.N. was 8 min. 55·9 sec. to be added to apparent time, and decreasing 0·7 sec. per hour.

The first step is to find the equation of time at the moment of the observation, about $17\frac{1}{2}$ minutes *after* G.M.N.

		min.	sec.	
Eqn. of time at G.M.N.	=	8	55·9	
Decrease in $17\frac{1}{2}$ min. $= \frac{17{\cdot}5}{60} \times 0{\cdot}7 =$			0·2	
Equation of time	=	8	55·7	To be added to app. time.

Now as the sun is on the meridian, it is local apparent noon.

		hr.	min.	sec.	
In other words, L.A.T.	=	12	0	0	
To find L.M.T., eqn. of time	=		8	55·7	
∴ L.M.T.	=	12	8	55·7	
Now G.M.T.	=	12	17	26	by chronometer.
Diff. = longitude in time	=		8	30·3	

The longitude is west, as the Greenwich time is " best."

To reduce it to angle we must multiply by 15.

8 min. of time $= 15 \times$ 8 min. of angle	=	2°	0′	0″	
30·3 sec. ,, $= 15 \times$ 30·3 sec. ,,	=		7	34·5	
∴ Longitude	=	2	7	34·5	W.

(2) At what Greenwich mean time should the sun be on the meridian in longitude 4° 24′ west, on 16th Oct., given that the equation of time at G.A.N. is 14 min. 15·1 sec. to be subtracted from apparent time and increasing 0·5 sec. per hour?

Here the Greenwich mean time of the observation is to be found, so we cannot begin by finding the equation of time, as in the last exercise.

We know, however, that the longitude is $4^\circ\ 24'$ W., and, reducing to time, by dividing by 15,

$$\begin{aligned} 4^\circ &= \tfrac{4}{15} \text{ hr.} &&= 16 \text{ min.} \\ 24' &= \tfrac{24}{15} \text{ min.} &&= 1 \text{ min. } 36 \text{ sec.} \\ &\quad\text{Total} &&= 17 \text{ min. } 36 \text{ sec.} \end{aligned}$$

Hence apparent noon at the place will be 17 min. 36 sec. after apparent noon at Greenwich.

Hence we take out the equation of time at Greenwich *apparent* noon, and correct *that* for this interval.

$$\begin{array}{llrr} & & \text{min.} & \text{sec.} \\ \text{Equation of time at G.A.N.} & = & 14 & 15{\cdot}1 \\ \text{Variation in } 17{\cdot}6 \text{ min.} = \dfrac{17{\cdot}6}{60} \times {\cdot}5 & = & + & 0{\cdot}1 \\ \hline \text{Eqn. of time} & = & 14 & 15{\cdot}2 \end{array}$$

to be subtracted from apparent time.

$$\begin{array}{llrrr} & & \text{hr.} & \text{min.} & \text{sec.} \\ \text{Now L.A.T.} & = & 12 & 0 & 0 \\ \quad\text{Eqn. of time} & = & - & 14 & 15{\cdot}2 \\ \hline \quad\text{L.M.T.} & = & 11 & 45 & 44{\cdot}8 \\ \text{Longitude corr.} & = & + & 17 & 36 \\ \hline \quad\text{G.M.T.} & = & 12 & 3 & 20{\cdot}8 \end{array}$$

Here the equation of time is subtracted because the *Nautical Almanac* states that it is to be *subtracted from apparent time*; the longitude correction is added because we are finding *Greenwich time*, and as longitude is west, Greenwich is "best."

The method of working here given is not mathematically exact, because the interval of 17 min. 36 sec. (as above) between G.A.N. and L.A.N. is in *apparent* hours. Now the variation per hour in the equation of time is per *mean* hour, and this is not quite the same as an apparent hour.

The greatest difference in length between a mean hour and an apparent one is, however, only $1\frac{1}{4}$ sec., and it is always under 1 sec., except from early December to early January. It is, in fact, represented by the variation per hour in the equation of time. If we take 1 sec. the variation in the equation of time in an *apparent* hour will differ from that in

one *mean* hour by $\frac{1}{3600}$th part. Hence the error is quite negligible except for the most refined work.

Reasons for Difference.

We must now try to explain the reasons why a solar day is not always the same length. For this purpose, we must

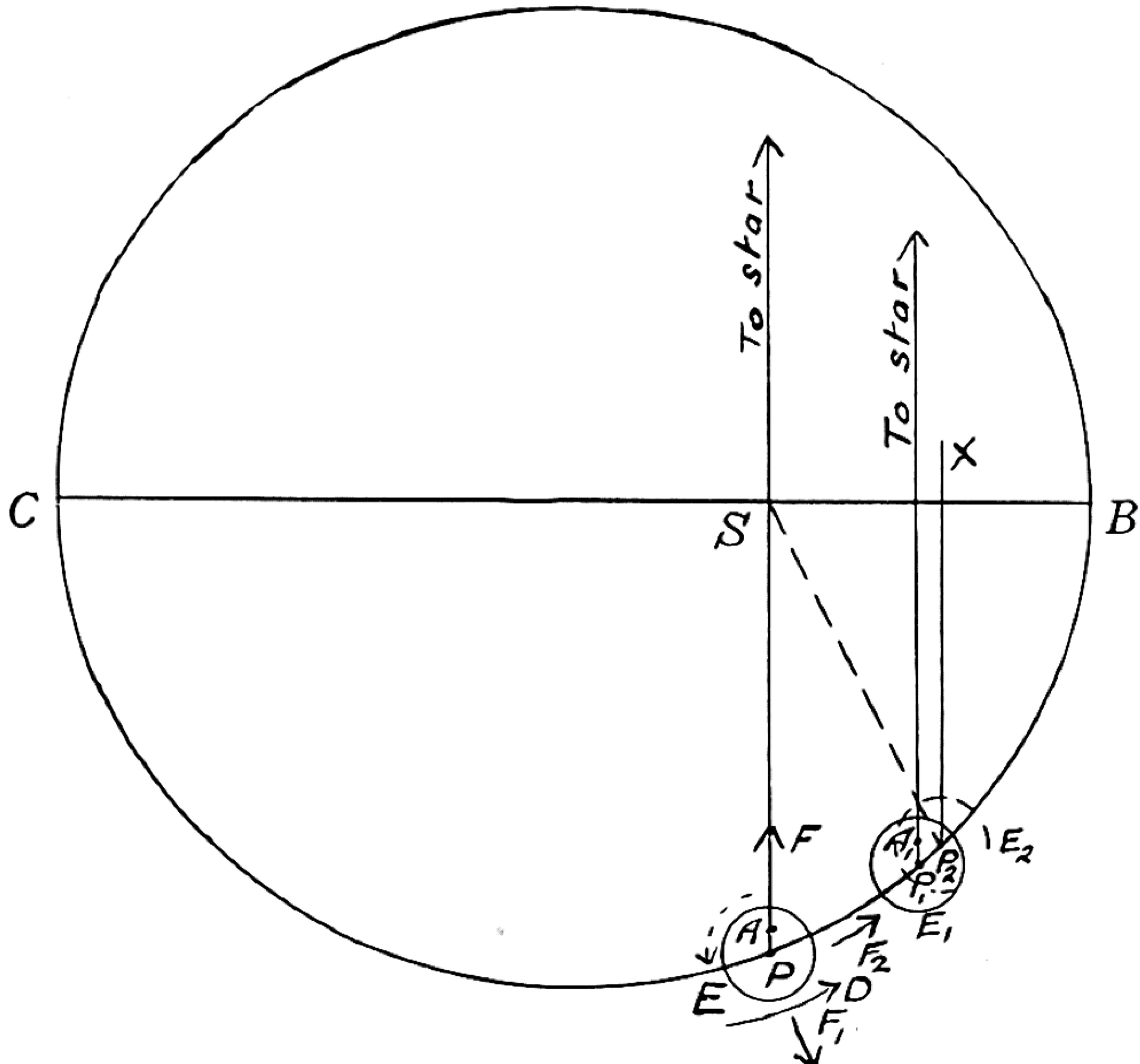


Fig. 9

temporarily abandon the idea of a moving celestial sphere and a fixed earth, and revert to the actual facts.

It is well known that the earth moves round the sun in an orbit which is not circular but *elliptical* in shape. The sun is not at the centre of this orbit, but at one focus of the ellipse.

In Fig. 9, let S be the sun, E the earth, and suppose the arrow D shows the direction of movement.

The earth is acted upon by two opposing forces, F, F_1, namely, the gravitational attraction of the sun, drawing it *inwards*, and the centrifugal force due to its motion in a curved path and tending to move it outwards. The latter depends upon the radius of curvature of the earth's path and upon the velocity of movement. These two, as the figure shows, do not exactly balance one another, and give a small resultant F_2, which at the point marked in the earth's orbit, is *in the direction of motion.* This causes the earth to move faster in its orbit as it approaches the point B, where it is nearest to the sun.

After passing B, the effect is reversed, and the rate of motion gets slower until the earth reaches the point C, where it is farthest away. Thus the angle which the earth describes round the sun per day is variable, and has its maximum value when the earth is at B. The fact that the distance BS is also least then increases the effect of the more rapid movement.

Now suppose that P is the North Pole of the earth, and A a station, so that PA is the meridian of the station. And suppose we start at a moment when this meridian passes through the sun's centre. It is then *apparent noon* at A.

Now besides moving round the sun, as shown by the full arrow, the earth is rotating much more rapidly, round its own axis, in the same direction, as shown by the dotted arrow.

When it makes one complete rotation, the meridian PA (which rotates with it, of course) will have described 360° from its original position, and will have come back parallel thereto, as shown at P_1A_1. But in the meanwhile the earth has moved round the sun from E to E_1, so that, as the diagram shows, the meridian P_1A_1 will *not* be again through the sun's centre. In order that the meridian may return to the sun, the earth must move on to the dotted position, E_2, while, in the same time, it rotates round its own axis, through the additional angle, XP_2S, required to bring the meridian back to the sun. The interval between these successive transits of the sun is one *apparent solar day*. It is clear that it is always longer than the time required for the earth to make one complete rotation.

Now the extra angle to be picked up is, as we have said, XP_2S, where XP_2 is parallel to SP. Hence $XP_2S = PSP_2$, which is the angle moved by the earth round the sun in one day.

Now, as we have seen above, this angle is variable.

Hence the extra angle to be described is also variable. And as the time taken by the earth to make one rotation on its own axis is, so far as we know, constant, it is clear that the length of the apparent solar day will be variable, and will have its greatest value about the end of December, when the earth is at *B*, as the angle described round the sun daily is then at its maximum value. The *least* length of the solar day will occur about the end of June, when the earth is at *C*.

The plane of the paper in Fig. 9 is the plane of the *ecliptic*, that is, the plane in which the earth moves round the sun.

We have shown the earth as if its polar axis were perpendicular to the ecliptic, so that the meridian could be represented as a line in the figure.

As a matter of fact, it is well known that this is not so, the polar axis being inclined to the ecliptic at about $66\frac{1}{2}°$ instead of $90°$.

It can be shown that this fact would also lead to variations in the length of the solar day, even if the orbit were a true circle with the sun in the centre.

The reasoning in this case, however, is less simple, and we have said enough to show that the length *does* vary.

The actual length at any time of the year will be subject to the variations from *both* the above causes, and the combined result of the two decides the length of the day.

The *mean solar day* is the average length of the apparent day throughout the year.

The student will see, then, that to tell the exact position of the sun with respect to the meridian of a station, we must know the *local apparent time*.

Local *mean* time differs from this by the equation of time. *Civil* time is usually neither local mean time nor apparent time. It is the local mean time at some selected station, and is adopted as standard time over a considerable area, say Great Britain.

But none of these will enable us, at once, to say how any *star* (other than the sun) is situated, at that moment, with respect to the meridian.

Solar and Sidereal Time.

It becomes necessary, therefore, to study another kind of

time which *will* give us this information, and which is called *sidereal time*.

The stars being scattered through space all round the sun, we may suppose that at any moment there is a distant star which is on the meridian of a station at the same moment as the sun, though of course invisible.

Now going back to Fig. 9, suppose there is such a star just on the meridian PA at the same time as the sun, but very much farther away.

When the earth makes one complete rotation so that the meridian P_1A_1 is parallel to its old position, the star, being practically infinitely distant, *will* be again on the meridian.

Thus the interval between two successive transits of the same distant star *is* equal to the time taken by the earth to make one complete rotation.

It is called a *sidereal day*, and is clearly shorter than a solar day.

The *amount* by which it is shorter is the time taken by the earth to describe the extra angle XP_2S, which, as we have seen, is equal to PSP_2, the angle described round the sun in one solar day.

This angle varies from day to day, but in *one year*, of 365·2422 solar days, the earth moves 360° round the sun. Hence, on the average, it describes an angle of $\frac{360}{365{\cdot}2422}$ degrees per day. And the length of a *mean solar day* is greater than a sidereal day by the time taken to rotate through this angle.

Now the earth rotates through 360° in one sidereal day

Hence to rotate through $\frac{360}{365{\cdot}2422}$ degrees, it will require $\frac{1}{365{\cdot}2422}$ of a sidereal day.

Hence one mean solar day is greater than a sidereal day by $\frac{1}{365{\cdot}2422}$ of the latter.

$$\text{That is, 1 solar day} = 1 + \frac{1}{365{\cdot}2422}$$

$$= \frac{366{\cdot}2422}{365{\cdot}2422} \text{ sidereal days.}$$

Hence 365·2422 solar days = 366·2422 sidereal. That is, there are 366·2422 sidereal days in the year; or the stars

gain *one day in the year* on the sun, or, roughly, *two hours per month*, or nearly 4 min. per day.

As each kind of day is divided into 24 hours, and subdivided to minutes and seconds, it follows that 1 mean solar hour $= \frac{366 \cdot 2422}{365 \cdot 2422}$ sidereal hours, and so on.

		hr.	min.	sec.	
Thus, 1 mean solar hour	=	1	0	9·86,	sidereal
1 ,, ,, minute	=		1	0·16	,,
1 ,, ,, second	=			1·0027	,,
Conversely, 1 sidereal hour	=		59	50·17	mean solar
1 ,, minute	=			59·84	,, ,,
1 ,, second	=			·9973	,, ,,

Tables for the rapid conversion from one set of units to the other are given in the *Nautical Almanac*, under the head of "Time Equivalents, Tables of."

Thus a star which is on the meridian at midnight to-night will have passed the meridian before midnight to-morrow night, and will cross at about 10 p.m. in a month's time.

A clock which keeps pace with the distant stars, and records the same hour each night when the same star is crossing the meridian, is called a sidereal clock.

It is set to agree with the apparent solar time at the moment of the vernal equinox, when the sun's centre is just crossing the equator near the end of March.

Thus at the March equinox *apparent solar time and sidereal time agree.* But from that time forward sidereal gains on solar about 2 hr. per month, or more exactly, as above described, 9·86 sec. per hour. Approximately this may be taken as 10 sec. per hour.

The *Nautical Almanac* gives (for the meridian of Greenwich) the sidereal time daily when it is mean noon. Greenwich and local sidereal time are abbreviated to G.S.T. and L.S.T. respectively.

Examples.

We shall now give a few exercises to illustrate the use of these tables.

(1) Find the Greenwich sidereal time on 16th June, 1926, when the Greenwich mean time was 9 hr. 47 min. a.m.

The first step is to find how far away—in time—the given moment is from *mean noon*, because it is at G.M.N. that the *Almanac* tells us the sidereal time.

Now 9 hr. 47 min. a.m. is 2 hr. 13 min. before mean noon. But this is a mean time interval, or measured in *mean solar* hours, *etc.*, and we must convert them to *sidereal* hours, etc., before we can add to (or subtract from) the sidereal time at noon.

By tables in the *Nautical Almanac* or the figures on page 25, we have

	hr.	min.	sec.
2 hr. =	2	0	19·7
13 min. =		13	2·1
	2	13	21·8

Hence, expressed in sidereal hours, etc., the given moment is 2 hr. 13 min. 21·8 sec. *before* G.M.N.

	hr.	min.	sec.
But *at* G.M.N., from *Almanac*, G.S.T. =	5	35	59·3
Less	2	13	21·8
G.S.T. required =	3	22	37·5

(2) Find the G.M.T. when the G.S.T. was 23 hr. 57 min. 12 sec. on 1st Dec., 1926.

As before, the first step is to find the interval between the time of observation and mean noon.

	hr	min.	sec.
G.S.T. of observation =	23	57	12·0
From *Naut. Alm.*, G.S.T. at G.M.N. =	16	38	20·5
Observation is *after* noon by	7	18	51·5

This is expressed in sidereal hours. We must reduce them to mean time.

	hr.	min.	sec.
7 hr. =	6	58	51·2
18 min. =		17	57·1
51·5 sec. =			51·4
G.M.T. =	7	17	39·7 p.m.

Sidereal Noon.

We have said that the sidereal clock is set to agree with the apparent solar time at the vernal equinox. This is purely a conventional arrangement, but, in consequence of it, the position of the sun's centre in the heavens at the moment of the spring equinox, is given a special name. It is called the " First Point of Aries " (because many centuries ago, when the name was given, the sun was on the edge of the constellation Aries at the equinox).

It is *sidereal noon*, then, when the first point of Aries is on the meridian. It is apparent noon when the sun's centre is there. At the moment of the equinox, the sun's centre and the first point of Aries are one and the same point, so that solar and apparent time agree.

But the student should remember that whereas a solar clock usually registers 12 o'clock at or about solar noon, a sidereal clock always registers zero when it is sidereal noon, and goes round to 24 hours at the next sidereal noon.

At the time of the equinox, mean time is about 7 or 8 minutes ahead of apparent time, so that the mean time is about 12 hr. 7 min. p.m. when it is sidereal noon, or the sidereal time is about 23 hr. 53 min. when it is mean noon.

Thereafter the sidereal time at mean noon increases by about 4 minutes daily.

Thus, without the *Almanac*, we can always find the sidereal time, roughly, at any given solar time and date.

Example.

(1) Find the sidereal time at 7 p.m. on 23rd Feb.

Feb. 23rd is about one month *before* the spring equinox. Hence sidereal time is about 2 hours *behind* solar, at the rate of 2 hours per month (p. 25).

The solar time is 7 hours after noon.

Hence sidereal time = 5 hours roughly.

We must not say either a.m. or p.m. in connection with sidereal time.

(2) Find at about what hour, solar time, the sidereal time will be 17 hr. 20 min. on 9th Oct.

Oct. 9th is about 6½ months *ahead* of the March equinox. Hence sidereal time is about 13 hours *ahead* of solar.

Therefore the solar time will be about 4 hr. 20 min. p.m. Of course this is only a *rough* rule.

Right Ascension.

Let us return now to the idea of the celestial sphere, on which the stars are supposed to be placed, and which is supposed to rotate round the fixed earth.

The first point of Aries is fixed on this sphere by the position of the sun's centre at the equinox. The sun is then on the equator. Hence, so is the first point of Aries.

Thereafter, the sun would appear to move *backward* (i.e. west to east) on the stellar sphere (in consequence of the gain in sidereal time as compared with solar), making the complete circuit in the year. But the first point of Aries remains as a fixed point on the celestial sphere, and sidereal time is measured by it.

Now, as we have seen, each star has its angular distance from the equator tabulated under the head of *declination*.

But to fix its position on the stellar sphere completely, we must have another co-ordinate, corresponding to terrestrial *longitude*, just as declination corresponds to latitude.

For this purpose, the meridian of the first point of Aries is chosen as standard, just like Greenwich on the earth for longitudes.

The angular distance of any star's meridian from this standard meridian is called the *right ascension* of the star, and is tabulated along with the declination.

But whereas longitudes on the earth are measured *east or west* of Greenwich, right ascensions are measured eastwards only, and go right round from zero to 360°, instead of 180° each way.

Moreover, instead of being expressed in degrees, like longitude, they are, for convenience, expressed in *time*, at the rate of 1 hour to 15°, or 24 hours to 360°.

Thus right ascensions vary from zero to 24 hours, and are *eastwards* from the first point of Aries (i.e. *opposite* to the direction of rotation).

Now several important results follow from this, which the student must learn.

First, in Fig. 10, suppose we are looking down on the celestial sphere from above the North Pole, *P*.

Let Z be the zenith of a station, so that PZ is the meridian of the station, and let S be a star, just crossing the meridian. Let F be the position, at the same moment, of the first point of Aries.

Then, when F was on the meridian, it was sidereal noon, and the sidereal time increases thereafter at the rate of 1 hr. per 15° of rotation of the sphere.

If the arrow shows the direction of rotation, ZPF is the angle turned through since F crossed the meridian.

Hence ZPF, reduced to time at 1 hr. to 15°, gives the *sidereal time* at the moment considered.

But from what we have said, the angle FPS, reduced to time, is the star's *right ascension*.

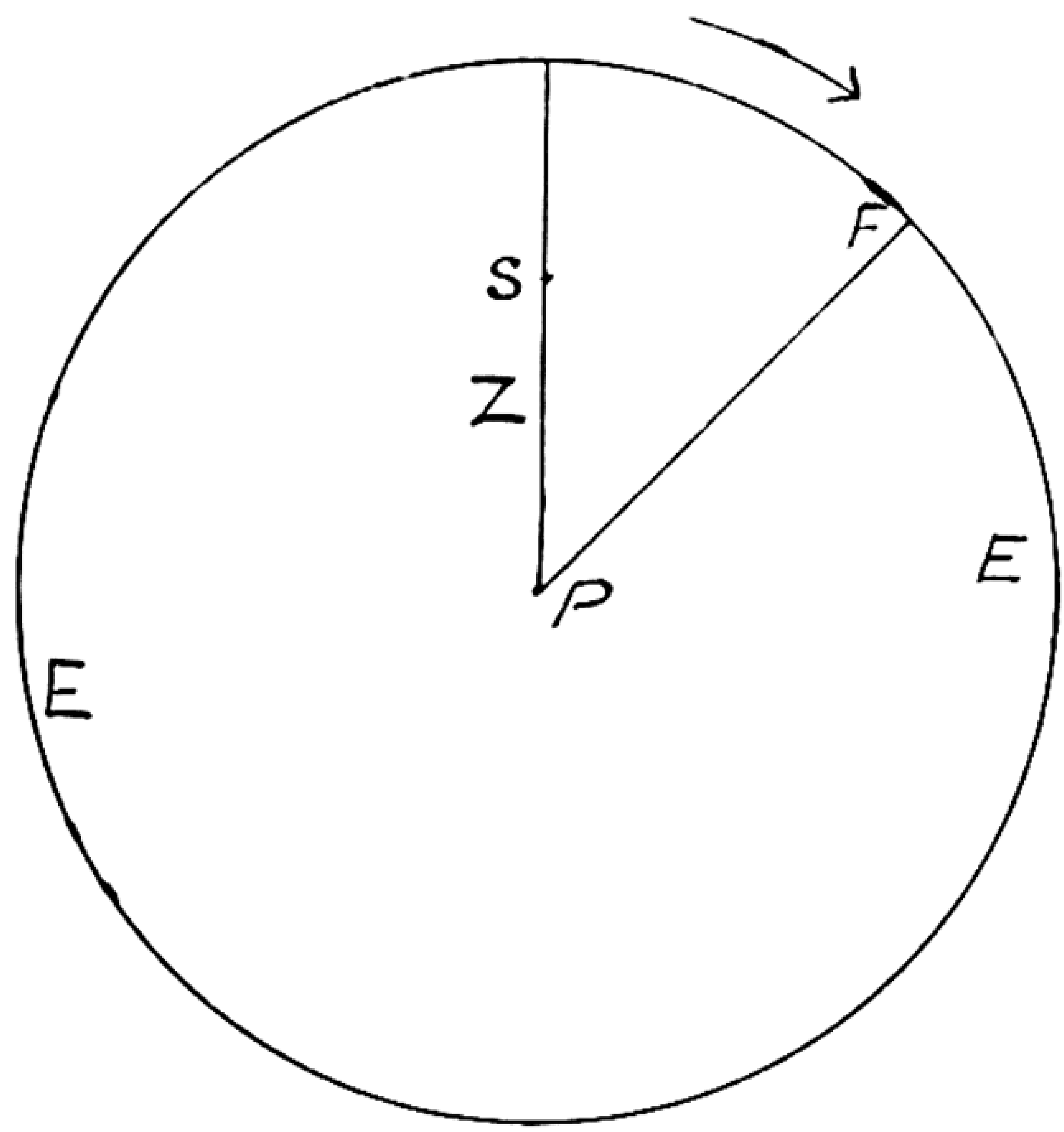


FIG. 10

Now when S is on the meridian, clearly $ZPF = FPS$. Hence we have the important result that *when a star is on the meridian of a station the local sidereal time is equal to the star's right ascension.*

In the *Nautical Almanac*, the bright stars are tabulated in order of Right Ascension. (See table on p. 9.)

Examples.

(1) If observation for latitude is to be taken on 7th Oct., and the desired time is about 7 p.m., find what star will be suitable, the latitude being approximately 50° N.

The first step is to find the sidereal time.

Oct. 7th is about 6½ months ahead of the March equinox. Thus sidereal time is about 13 hr. of solar, and therefore will be about 20 hr. at 7 p.m.

Now for latitude observations the star must be on the meridian. And when it is there the sidereal time is the same as its right ascension.

Hence we must choose a star whose right ascension is about 20 hr., and whose declination is sufficient to enable it to be seen, preferably fairly high up.

Reference to the table on page 9 or to a *Nautical Almanac* or a star chart, shows that, among bright stars, there are Altair at 19 hr. 47 min.; γ Cygni at 20 hr. 20 min.; and Deneb at 20 hr. 39 min.

The last two would be very high up, and probably the first (that is Altair) would be best, but the observation would have to be made before seven.

We should try to set on the star soon after 6.30, and *follow* it, as described on page 13.

(2) We shall now work the same example more exactly. Suppose that the station is in longitude 3° 37′ W. about, and the year 1926, and that the time 7 p.m. refers to Greenwich mean time.

		hr.	min.	sec.
Now on 7th Oct., 1926, G.S.T. at G.M.N	=	13	1	30
Converting the solar interval to sidereal, 7 hr.	=	7	1	9
G.S.T.	=	20	2	39
Longitude corr.	–		14	28
L.S.T.	=	19	48	11

The longitude correction is arrived at as already stated. The result shows that Altair will be almost exactly right, and we need not set on the star until about 6.50 or 6.55, according to how sure we are of the time.

(3) Find the longitude of a station if Aldebaran crossed the meridian at 8 hr. 12 min. 12 sec. p.m. by G.M.T. chronometer on 3rd Feb., 1926.

Here we must find the G.S.T. from the G.M.T.

Local sidereal time is given by the star's right ascension.

Thus we shall know both Greenwich and local time, ***in the same kind of time,*** and the difference gives longitude.

		hr.	min.	sec.
G.S.T. at G.M.N. on 3rd Feb., 1926	=	20	51	37·5
8 hr. 12 min. 12 sec., *after* noon, 8 hr.	=	8	1	18·9
12 min.	=		12	2·0
12 sec.	=			12·0
G.S.T.	=	29	5	10·4

Clearly we have gone past the next sidereal noon and we must subtract 24 hr., giving

		hr.	min.	sec.	
		5	5	10·4	
L.S.T. = Star's right ascension	=	4	31	40·3	
Longitude	=		33	30·1	W.

33 min.	$= 15 \times 33'$	=	8°	15	0″	
30·1 sec.	$= 15 \times 30{\cdot}1''$	=		7	32	
Answer		=	8°	22′	32″	W.

More worked examples will be given in Chapter V, and at the end of the book there are examples for the student's exercise, with answers.

Azimuth.

From those which have been given, it is clear that such calculations may be made for two different purposes—

(1) To find a suitable star for a latitude observation, and to find when it should be observed ;

(2) To determine longitude by taking the time when a star crosses the meridian.

For the first of these purposes, it is not necessary (though it is helpful) to know the direction of the meridian at all accurately. We can observe the star a little before time, and *follow* it to find its highest altitude.

For the second purpose, however, we wish to know the *exact moment* of crossing, and are not concerned with altitude at all. Hence we must know the exact direction of the meridian.

That is to say, before we can take an exact observation for

longitude, we must know the exact direction of a north and south line at the station.

This is found by determining the horizontal angle at the station between some known line on the ground and the meridian.

The best methods of doing this require a knowledge of spherical trigonometry for their solution.

By Equal Altitudes.

We shall now describe a simple method, known as the method of equal altitudes.

If the altitudes of a star be observed at a number of known times before and after it crosses the meridian, and the results

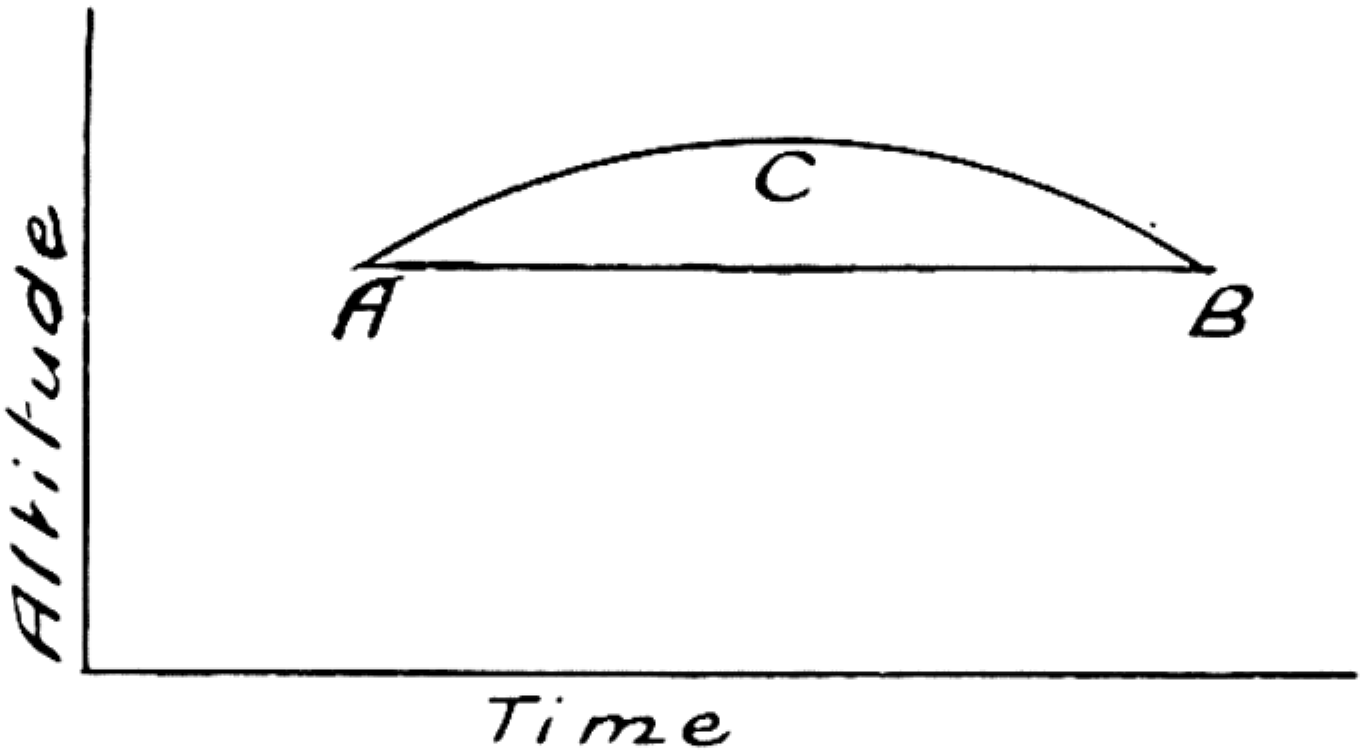


FIG. 11

are plotted in a graph, we obtain a curve such as that shown in Fig. 11. Assuming that there has been no appreciable change of declination, and that the observations are correct, the curve will be symmetrical about the meridian.

The highest point C will correspond with the moment of transit, and, if A and B are two points where the altitudes are equal, C will be midway between these in time. It is evident from symmetry that if the *horizontal* angles were also read, the angle corresponding to the point C would also be midway between those read at A and B.

The method, then, is as follows: Set up the theodolite at the station, and level it very carefully, using the upper level. Take the reading of the horizontal circle to some fixed point on the earth, called a referring object, say another station.

Now choose a suitable star which is rising, and is at least

1 hr. away from the meridian. Set the cross hairs on it, both circles being clamped securely and the final setting made by tangent screws. Read the horizontal and vertical circles.

We must now wait for two hours or more until the star has crossed the meridian and returned to the *same altitude when setting*. When the proper time approaches, we turn the theodolite *horizontally* round towards the same star, and aim at it as carefully as we can, but taking care not to touch the vertical circle.

Presently the star will appear in the field of view. As it is setting, it will appear to be *rising*, and should come into view at the bottom of the field. We just turn the telescope horizontally to keep the star in view until it is approaching the horizontal hair. Then bring it near the vertical hair also, and clamp the horizontal motion. Use the *horizontal* tangent screw to keep the star accurately on the vertical hair, until, at last, it comes to the intersection of the two hairs, when the observation ceases.

Read the horizontal and vertical circles again. The vertical circle is read only as a check, to make sure that the altitude is unaltered. It should be the same as before. Hence only *one* vernier need be read.

But both horizontal verniers should be read for horizontal angles in all cases, and the mean taken.

Suppose the following results are obtained—

Name of station, K; referring object, L.

Horizontal reading to L,		217° 23′ 25″.	
,, ,,	star,	69° 48′ 30″,	1st position.
,, ,,	,,	162° 9′ 52″,	2nd ,,

∴ Mean reading to star = 115° 59′ 11″, and this gives the reading when the theodolite is pointing due south (or due north, as the case may be), that is, along the meridian.

Thus, reading to L from K	=	217°	23′	25″
Reading along meridian	=	115	59	11
Angle between KL and meridian	=	101	24	14

This angle is *clockwise* from the meridian to KL, because the theodolite reads clockwise, and the meridian reading is the smaller.

Assuming that our station K is in the northern hemisphere,

and that the star is observed to the south, then the meridian direction above is due south. Draw KN (Fig. 12) to represent north, KM south. Set off the angle 101° 24′ 14″ clockwise from KM, and we find the direction of the referring line KL.

Now the angle NKL, measured clockwise from north to the line KL, is called the *azimuth* of KL.

In this case it is clear that it is greater than MKL by 180°.

Hence we have

$$
\begin{array}{rcl}
MKL & = & 101^\circ\ 24'\ 14'' \\
 & & 180 \\
\hline
\text{Azimuth of } KL & = & 281^\circ\ 24'\ 14''
\end{array}
$$

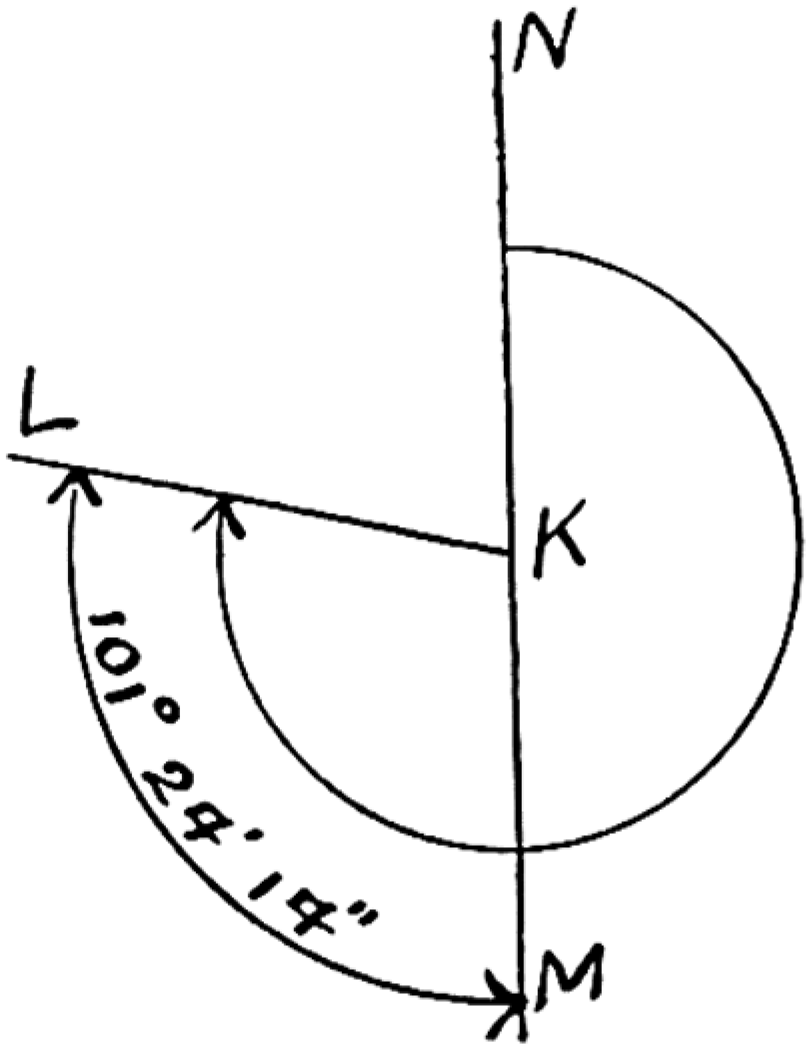


FIG. 12

This angle must be set off *anti-clockwise* from the line KL in the field, that the theodolite may point due north, or 101° 24′ 14″ in order that it may point due south.

The disadvantages of this method are, first, that so long a time is required; secondly, it is difficult to ensure that the instrument shall be levelled with sufficient precision to ensure exactly the same altitude; and, thirdly, that the result is affected by errors in the permanent adjustments of the theodolite, unless the star and the referring object are at the same altitude.

The time taken is not only a disadvantage in itself, but it may lead to changes in atmospheric refraction, which may affect the result.

The second objection can be minimized with care, but cannot be overcome. The third may, theoretically, be overcome by taking all observations both face left and face right. This, if done in one evening, means that we must read the referring object face-left and face-right to start with. Then set on the star, say, face-left, and read horizontal and vertical circles. Then reverse face immediately, and take the readings again on the star, face-right.

Then take the readings at the same altitude when setting,

still face-right, as before. Then reverse face immediately, set the vertical circle to the same reading as we had before, face-left, and wait for the star to cross. Means between face-left and face-right readings are taken in all cases.

The difficulty is to be sure that we set back to exactly the same altitude.

With Sun.

When finding south by morning and afternoon observations of the sun by this method, allowance must be made for change of declination.

Suppose we are making an observation on 1st March. At that time the sun's declination is *increasing* (i.e. he is moving towards the north), about 57″ per hour. Hence if we take our observation about $1\frac{1}{2}$ hr. before and after noon, the declination will have increased $3 \times 57'' = 3$ min. nearly between the observations.

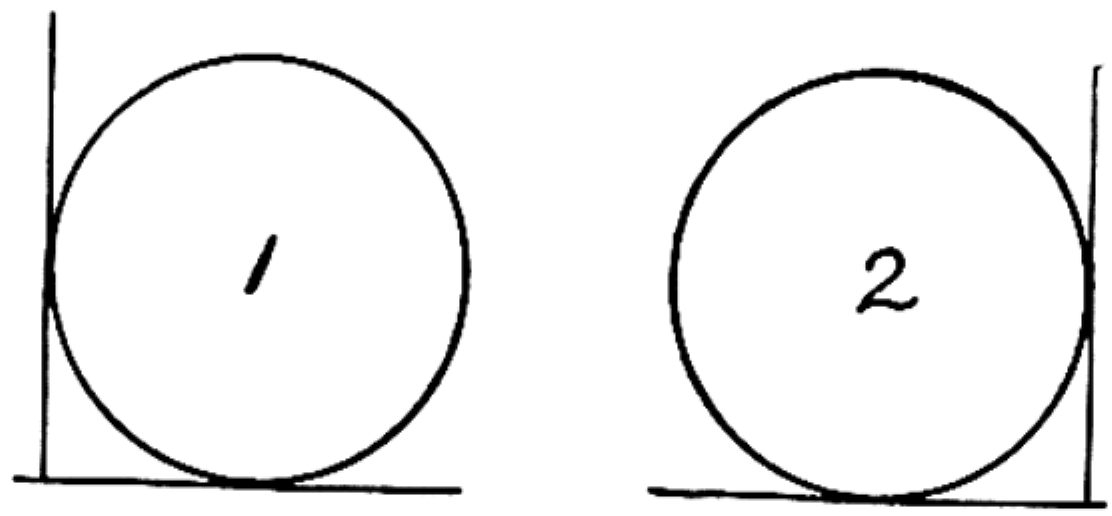


FIG. 13

Thus the sun will be about 3 min. higher up at the second observation than he would have been if his declination had not changed, and we should set the vertical circle to read 3 min. more for the second observation than for the first.

The change in altitude is not quite equal to the change in declination, but at $1\frac{1}{2}$ hr. from the meridian this will be near enough in most cases.

It is, moreover, necessary to read the horizontal angle to the *edge* of the sun, instead of the centre (p. 15). Care must be taken that the sun's image is either above *or* below the horizontal hair for *both* observations (for equal altitudes), but that it is *right* of the vertical hair for one, and *left* of it for the other, so that the mean horizontal reading will give the angle to the sun's centre (Fig. 13).

Examples.

(1) On 1st March in approximately longitude 5° W., if it is desired to observe the sun about 1 hr. before and after noon,

about what time must the first observation be taken, by a clock which gives G.M.T.?

The local apparent time is to be about 11 a.m. Greenwich mean time will be ahead of this by $\frac{5}{15}$ hr. or 20 min. Hence G.A.T. = 11 hr. 20 min., and the equation of time is about $12\frac{1}{2}$ min., to be added to apparent time, so the G.M.T. of the first observation should be about 11 hr. $32\frac{1}{2}$ min., say 11.30.

(2) It is desired to make an azimuth observation in England, about 6 to 8 p.m., on 20th Oct. Find a suitable star, and when the first observation should be made by G.M.T. chronometer, in long. 5° W.

The G.M.T. of observation is to be about 7 p.m. for transit.

The G.S.T. at noon is approximately 13 hr. 50 min.

The G.S.T. at 7 p.m. is about 20 hr. 50 min.

$\therefore$ L.S.T. = 20 hr. 50 min. − 20 min. = 20 hr. 30 min., and this will be the right ascension of a suitable star.

Reference to the *Nautical Almanac* shows a scarcity of bright stars about then, the nearest being γ Cygni at 20 hr. 19 min. 30 sec. about; but this one would be rather high up for convenient observation.

It might be better to use Altair (19 hr. 47 min.), in which case the first observation should be taken not much later than 5 p.m.

More exact methods of determining azimuth require a knowledge of spherical trigonometry, and are referred to in a later chapter.

Longitude by Wireless.

It will be seen that when the exact direction of the meridian is known, local time (either sidereal or apparent) can be found by taking the moment at which a star (or the sun) crosses the meridian. By many such observations, it can be found very accurately.

To find longitude we must also know the Greenwich time. If this is found by chronometer, it is always subject to some small uncertainty. Hence for the best results, telegraphic signals are arranged between the nearest station of known longitude and that whose longitude is required. Nowadays wireless signals are sent out regularly at certain times from certain stations.

By such methods, Greenwich time can be found with more

certainty than by chronometer. They are, however, refinements of method, and here we are chiefly concerned with principles, so that we shall not consider them farther.

Longitude by Moon Observations.

Where wireless signals are not available, and the Greenwich mean time chronometer is considered unreliable, *Greenwich* time may be found by direct observation of the moon or of certain phenomena connected with some planets. Such observations take many forms, but we propose here to refer to observations on moon culminating stars.

The right ascension of the moon changes rapidly. Hence if we can find it at some given moment by direct observation, the *Nautical Almanac* enables us to compute the *Greenwich* time corresponding thereto.

Moon culminating stars are stars whose right ascensions and declinations nearly agree with those of the moon on the day of observation, and hence are suitable for comparison.

Having found our meridian, we choose such a star, and observe the exact time at which it crosses the meridian, by any fairly accurate time-keeper, which need not, however, give true local or Greenwich time. The local sidereal time is then equal to the star's right ascension.

Now, according to the phase of the moon, one or other of its edges will be complete. Before full moon this will be the *right* or *leading* limb or edge ; after full moon, the right edge is in shadow, and the left complete. The observation of the moon must be taken on the complete edge, whichever it is, and we take the exact time of its transit.

The *difference* between the times for the star and moon gives the *time interval* between the transits. It should be small, so as to minimize the effect of any changes in the clock rate.

It is converted to a sidereal interval (unless a sidereal clock was used), and *added* to the star's right ascension, if the star transits first, or *subtracted* if the moon was first.

The result clearly gives the local sidereal time of the transit of the moon's edge, and this, in turn, is equal to the right ascension of the edge, or limb, at that moment.

Now the *Nautical Almanac* (under the heading " Moon, at transit at Greenwich ") gives us the right ascension of the complete limb when it crossed at Greenwich, and how much

it was changing per hour of longitude. Hence we can find the longitude. Strictly, we should allow for a variable rate of change, but here we regard it as constant.

Example.

On 24th Nov., 1928, O Piscium (right ascension, 1 hr 41 min. 38·2 sec.) crossed the meridian of a station at 9 hr. 32 min. 5·4 sec. p.m. by a mean time clock. The right limb of the moon crossed at 9 hr. 37 min. 13·0 sec.

Find the longitude, given that, at the transit at Greenwich, the right ascension of the moon's bright limb (the right) was 1 hr. 45 min. 23·1 sec., increasing 139·06 sec. per hour of longitude.

		hr.	min.	sec.	
Time of transit of moon's limb	=	9	37	13·0	
,, ,, star	=	9	32	5·4	
Mean time interval	=		5	7·6	
Sidereal interval	=		5	8·4,	moon later.
Right ascension of star	=	1	41	38·2	
,, ,, moon's limb	=	1	46	46·6	
At transit at Greenwich	=	1	45	23·1	
Increase	=		1	23·5	= 83·5 seconds.

But increase per hour of longitude (i.e. per 15°) is 139·06 sec.

$$\text{Hence, longitude} = 15^\circ \times \frac{83{\cdot}5}{139{\cdot}06} = 9^\circ \text{ almost exactly.}$$

The longitude is *west* because the moon's right ascension is *greater* than at Greenwich, and hence the transit is later.

Alternatively, we may reduce the observation to the moon's centre. Thus, in the above case, the *Nautical Almanac* tells us also that the time taken by the moon's semi-diameter to cross the meridian is 69·0 sec.

Hence we have—

		hr.	min.	sec.
Time of transit of right limb	=	1	46	46·6, L.S.T.
For semi-diameter add			1	9·0
Time of transit of centre	=	1	47	55·6,

and this gives the right ascension of the moon at transit.

Now the *Nautical Almanac* tells us that at 10 p.m. on that day, G.M.T., the moon's right ascension was 1 hr. 47 min. 34·4 sec., increasing 22·4 sec. in 10 min.

		hr.	min.	sec.
Right ascension at transit	=	1	47	55·6
At 10 p.m., G.M.T.	=	1	47	34·4
Increase	=			21·2

Time required $= 10 \times \frac{21{\cdot}2}{22{\cdot}4}$ min. = 9 min. 28·2 sec.

Hence, G.M.T. of transit = 10 hr. 9 min. 28·2 sec. p.m.

		hr.	min.	sec.	
Now Gr. sidereal time at noon	=	16	12	46·7	from *N.A.*
10 hr.	=	10	1	38·6	
9 min. 28·2 sec.	=		9	29·8	
Gr. sidereal time of transit	=	2	23	55·1	
Local ,, ,, ,,	=	1	47	55·6	as above
Longitude in time	=		35	59·5	
	=	9° almost exactly, as before.			

We can now find the clock error on local mean time, for

		hr.	min.	sec.
Gr. mean time of transit	=	10	9	28·2
Longitude correction	=	–	36	0
L.M.T.	=	9	33	28·2

	hr.	min.	sec.
But the observed local mean time of right limb was	9	37	13
Time of semi-diameter passing = 69 sec. sidereal, or, in mean time,		1	8·8
Observed time of transit of centre =	9	38	22

or the clock is nearly 5 min. *fast.*

The International Date Line.

It will be clear, from what we have said, that at any given moment the time depends on the longitude, increasing eastwards, and decreasing in the west.

Thus consider the moment when it is 9 p.m. at Greenwich, mean time, on 10th March.

In 45° E. the time will be 3 hr. ahead, and the local mean time will be midnight on 10th March; in 90° E. it will be 3 hr. ahead of that, and therefore 3 a.m. on 11th March, and finally at 180° E. it will be 6 hr. more ahead (as shown in Fig. 14), and therefore 9 a.m. on 11th March. But if we go

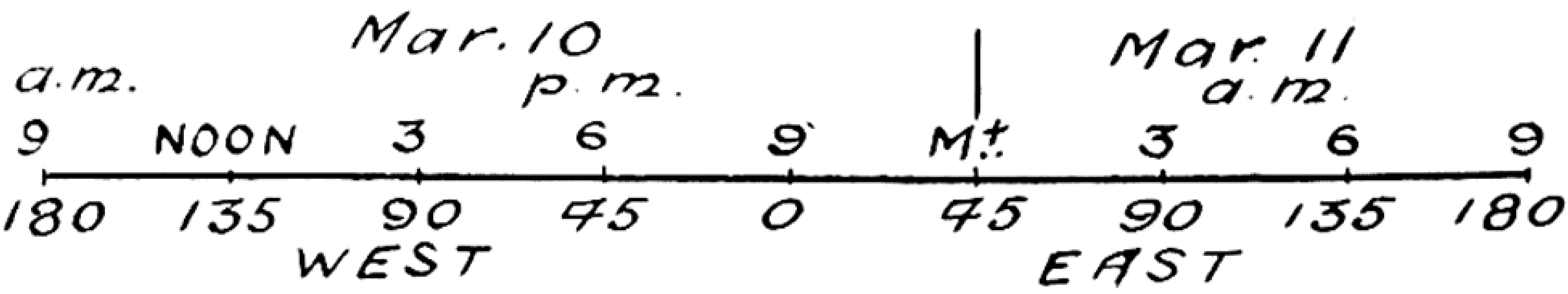


FIG. 14

westwards instead of eastwards, the time is behind Greenwich, so that, working westwards to 180° west, the time is 9 a.m. on 10th March, at the same moment as it is 9 p.m. on 10th March at Greenwich and 9 a.m. on 11th March at 180° E.

But the meridian of 180° W. is the same as 180° E.

Hence on this meridian the *time* will be 9 a.m. in any case, but the *date* may be either 10th March or 11th March, according to which way round we go.

If we start from any other point than Greenwich, the same thing will happen at the 180°th meridian.

Now it is clear that we cannot have a double date of this sort at a place, and it so happens that the meridian of 180° from Greenwich crosses land only in a very, very few places.

Hence, by general agreement, a line has been chosen which nearly follows this meridian but deviates slightly from it in places to avoid certain islands. This is called the international date line, and, in crossing it, the date is advanced, or put back, one day, according to whether we cross from west to east or east to west.

The line is shown in Fig. 15.

The 180° meridian is shown as a vertical straight line, full where it coincides with the date line, and dotted where the latter departs from it.

It will be seen that the line bends west, about 70° N. latitude, to avoid Wrangel Island. It then bends east through Bering Strait, then west again in order to keep the whole of the Aleutian group on one side of it. From about 50° N. to 10° S. it encounters no land and follows the meridian. It then bends east to keep Chatham Island and the whole of the Fiji group on the same side as New Zealand.

In the case we have been considering, the date would be 10th March in all the Aleutian Islands, but 11th March in the Fiji group.

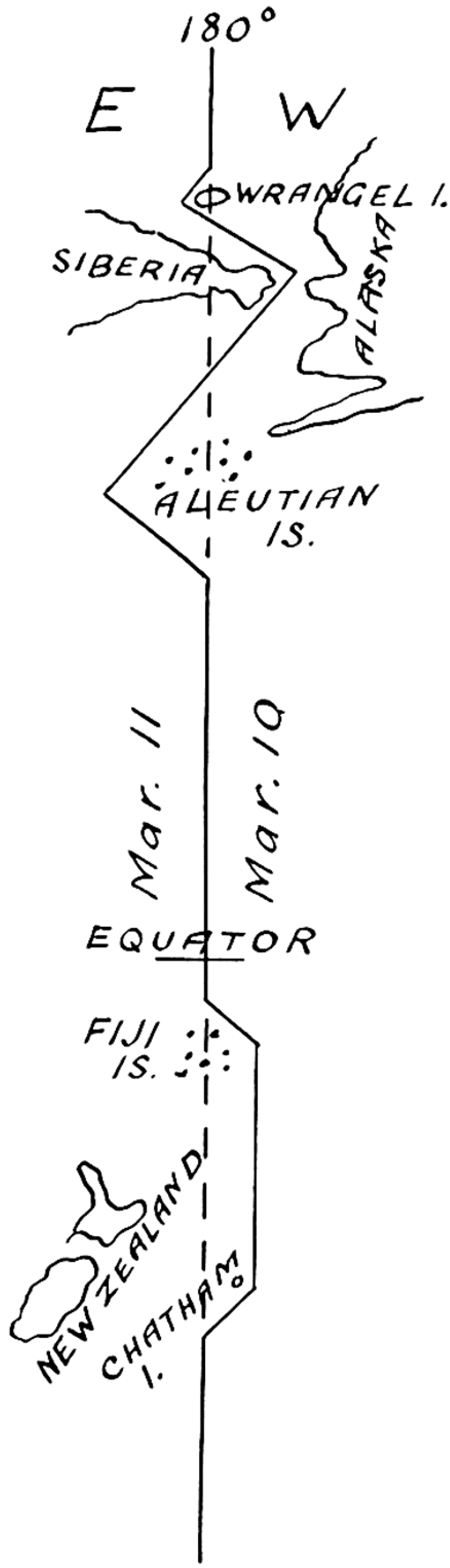


Fig. 15

Atmospheric Refraction

Alt. (Deg.)	Refr. (Sec.)	Alt. (Deg.)	Refr. (Sec.)
80	10	35	81
70	21	30	98
60	33	27	111
50	48	24	128
45	57	22	140
40	68	20	155

CHAPTER II

THE SHAPE AND SIZE OF THE EARTH

Historical.

FOR fuller notes of the history of the early efforts to determine the figure of the earth the student is referred to the article on this subject in the *Encyclopaedia Britannica*, but we must refer to it briefly here.

Everyone knows that (so far as existing records show) the first important attempt at determining the earth's radius was made by Eratosthenes, rather more than 200 years before the start of the Christian Era.

We have seen that if the meridian zenith distance (z) of a star be determined, and its declination is δ, then if latitude $= \lambda$, we have $\lambda = z + \delta$. . . . (p. 7).

Thus if the meridian zenith distances of the *same star* be observed at two different stations, it is clear that the *difference of latitude* between the stations will be the same as the difference of zenith distances, as δ is the same in both cases.

Now Eratosthenes noted that at the summer solstice the sun was directly overhead at Syene at noon, so that it cast no shadow from either side of a well there. In other words, the sun's meridian zenith distance at Syene was zero on that day.

He therefore measured the zenith distance on the same day at noon at Alexandria (which is nearly on the same meridian as Syene), and found it to be $7° 12'$.

He thus found that the difference of latitude between these places was $7° 12'$.

Now if A and B are two stations on the same meridian, and we assume (like Eratosthenes) that the earth is a true sphere, it is clear that the angle AOB (Fig. 16) is the difference of latitude.

Now $\dfrac{AOB}{360°} = \dfrac{\text{arc } AB}{2\pi r}$, where r is the radius of the earth.

Eratosthenes obtained in some way the distance from Syene to Alexandria as 5,000 stadia. This gives the arc AB.

But 360° is exactly 50 times 7° 12′; hence he found that 5,000 stadia was the fiftieth part of the earth's circumference.

The question of whether the result was good or not is of no importance. The facts that the angle is exactly the fiftieth part of the whole circle, and that the distance is given in round numbers, indicate what we should call a very rough determination, though it was an important scientific step.

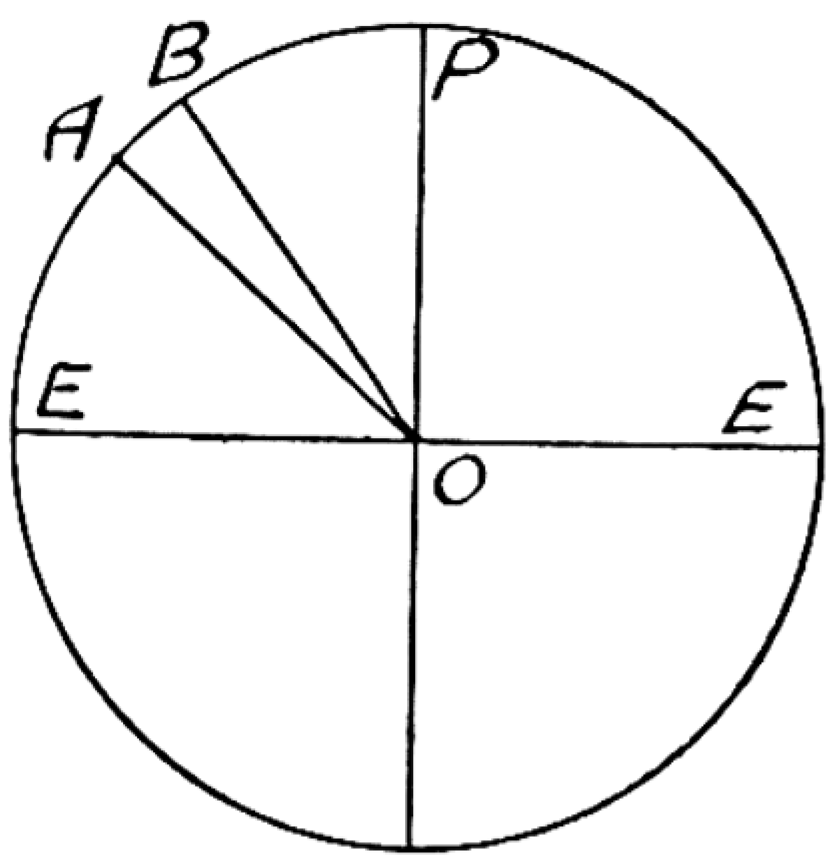


FIG. 16

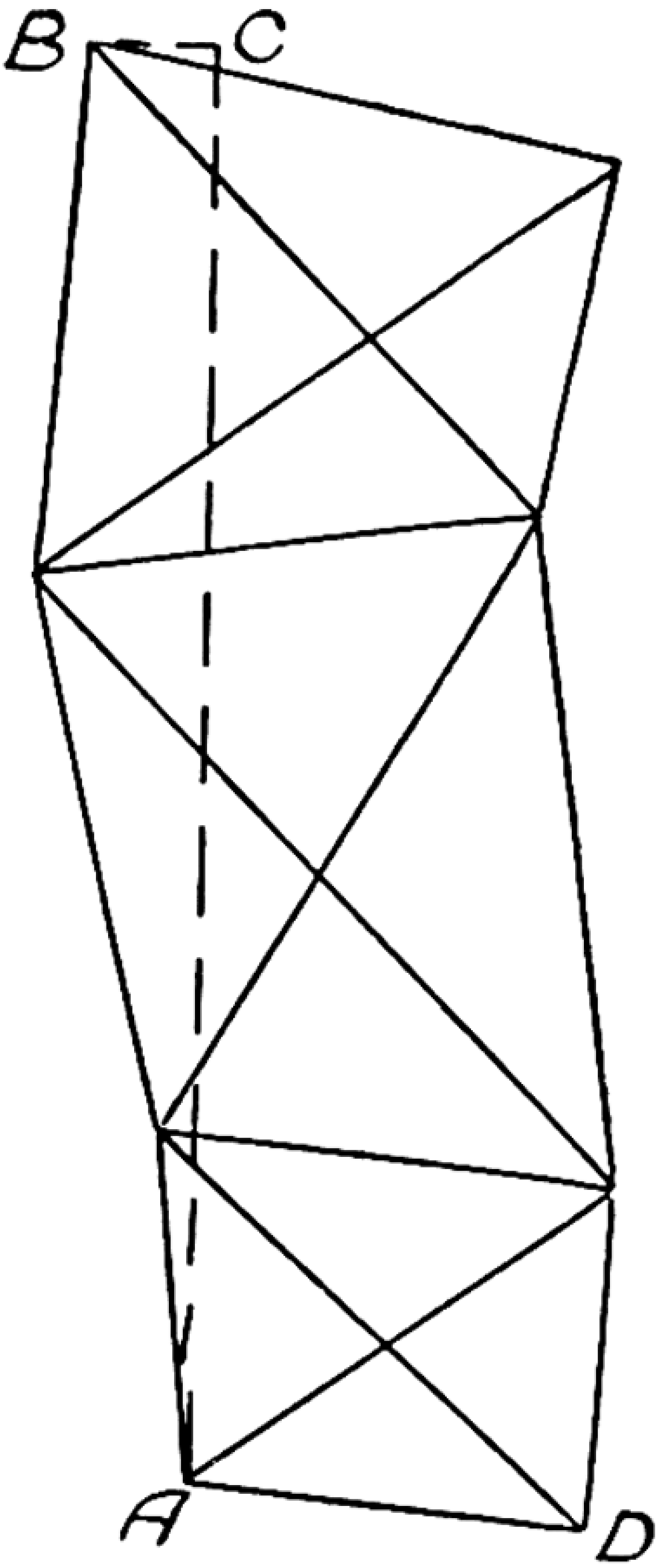


FIG. 17

The next important step was not taken until about the year 1617, when a Dutch geodecist, Willebrord Snell, obtained the distance between the terminal stations, not by direct measurement but by means of a chain of triangles linked up with a measured base. He introduced, in fact, the idea of applying a triangulation survey to the solution of this problem.

With such a chain as that shown in Fig. 17, connecting stations A and B, it is possible to use stations very much farther apart than would be possible by direct measurement, as well as obtaining the distance more accurately.

Knowing the azimuths of the lines as well as the latitudes of A and B, it is possible to calculate the distance AC, between A and B, in a *due north and south* direction, even if A and B are not exactly on the same meridian.

This is, in fact, the principle of the method now adopted for finding the radius of curvature of the earth at any point.

If $\triangle\lambda$ = difference between latitudes of A and B,

AC = north and south distance ; then

$$\frac{\triangle\lambda}{360^\circ} = \frac{AC}{2\pi r}$$

The next important advance was the application of the telescope in the measurement of the angles, made by Picard in 1669, in France.

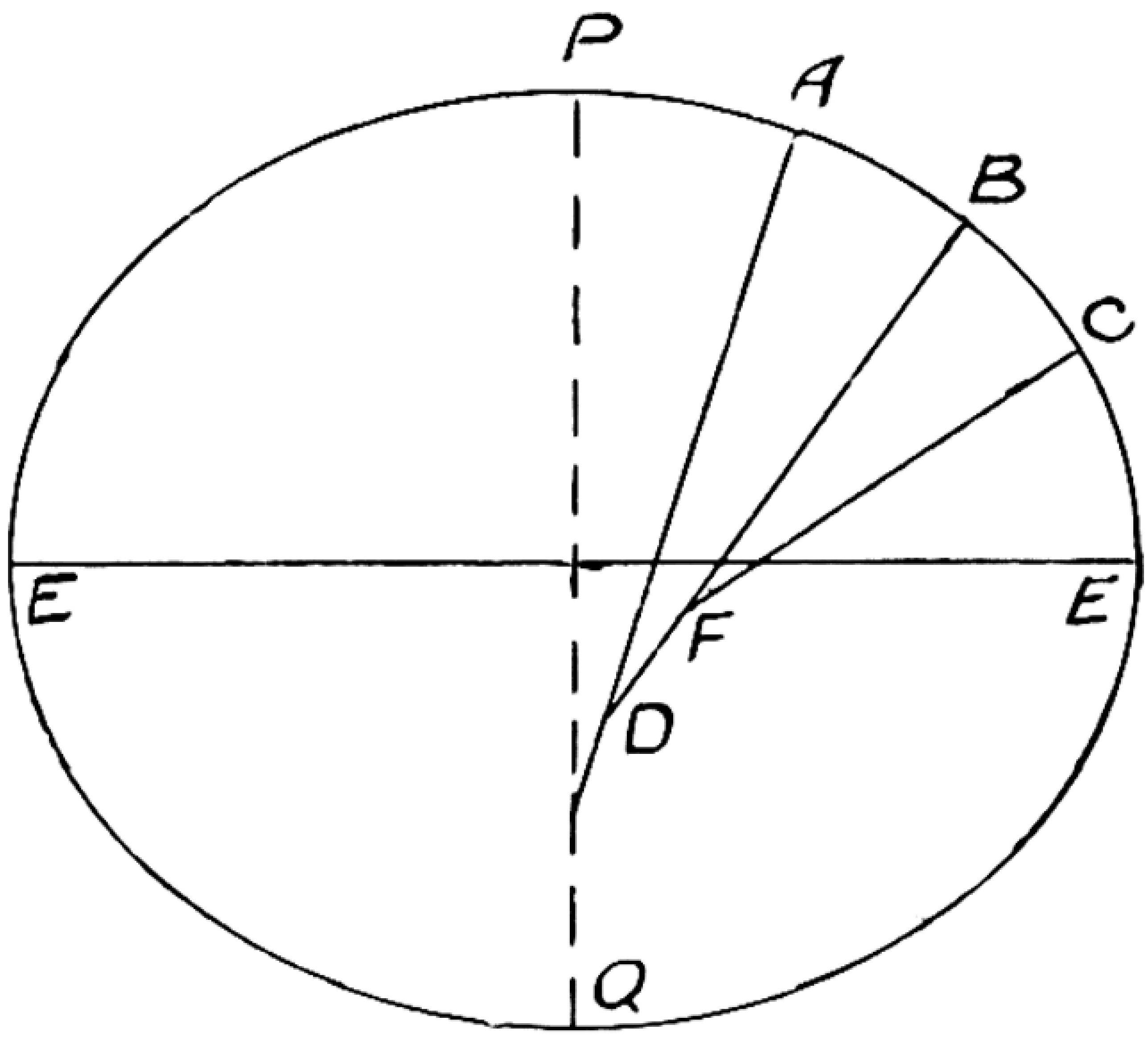


FIG. 18

This was soon followed by the remarkable discovery of Richer (1672) that a pendulum which beat seconds in Paris, correctly, did not do so at Cayenne, in South America, and the explanation (1687) of this fact by Newton, as due to the fact that the earth was *spheroidal* in shape, rather than spherical, and that the force of gravity would not, in consequence, be the same at different places.

Newton argued, on theoretical grounds, that the earth's polar axis should be shorter than its equatorial diameter in consequence of its rotation. Hence the nearer we approach the equator, the farther we should be from the centre, the smaller the force of gravity, and the longer the time taken by a pendulum to swing. Such a figure is called an *oblate spheroid*.

Now, referring to Fig. 18, if EE be the equator and PQ the polar axis, a section of the oblate spheroid would be an ellipse. The figure shown is very nearly a true ellipse, but has been described as a series of circular arcs with the centres marked. This makes it clear that, with such a shape, the radius of curvature along a meridian increases as we approach the

poles, and decreases towards the equator. Thus between A and B the radius is BD. But between B and C it is BF. The difference of latitude between A and B is the angle ADB; that between B and C is BFC. As the radius of curvature is greater for AB than for BC, it is clear that the length of arc, *per degree of latitude*, will also be greater. Thus, with this shape, we should find an increasing radius of curvature as we approach the poles, or, in other words, the length of one degree of meridian should increase with the latitude.

This was a matter which could be put to the test at once. The experiment was tried, and (after a preliminary trial which seemed to give a contrary result and created much excitement) the tests instituted by the French (1735 to 1745) completely established the validity of the reasoning of Newton, which had been strengthened by the researches of Huyghens.

This may almost be said to have led, later, to an alternative method of working at the earth's shape, by finding how the time of swing of a pendulum varies from place to place, and so arriving at a comparison between the strengths of the gravitational attraction at different places. Surveys on this principle are called *gravity surveys*, and will be referred to again later.

In 1783 a triangulation to connect the observatories of Greenwich and Paris was set on foot; improved methods were adopted both for the base line measurements and for the angles, and this, together with the French survey to determine the length of the metre (which was to be the forty-millionth part of the length of a complete meridional circumference), aroused considerable interest in the subject, so that Geodetic surveys were undertaken by many countries.

Another important step was the first application of the method of least squares to the adjustment of the angles by Gauss, in the Survey of Hanover (1821–44).

Since then many contributions to the solution of the problem have been made, among the most important of which has been the United States Coast Survey. In this the astronomical observations were corrected according to the theory of *isostacy* for the first time. In 1887 the International Geodetic Association was formed for the study of this and allied problems.

Mathematical Formulae.

The mathematics involved in the complete study of the spheroidal shape of the earth is somewhat advanced. It may be mentioned, however, that, assuming that the section along

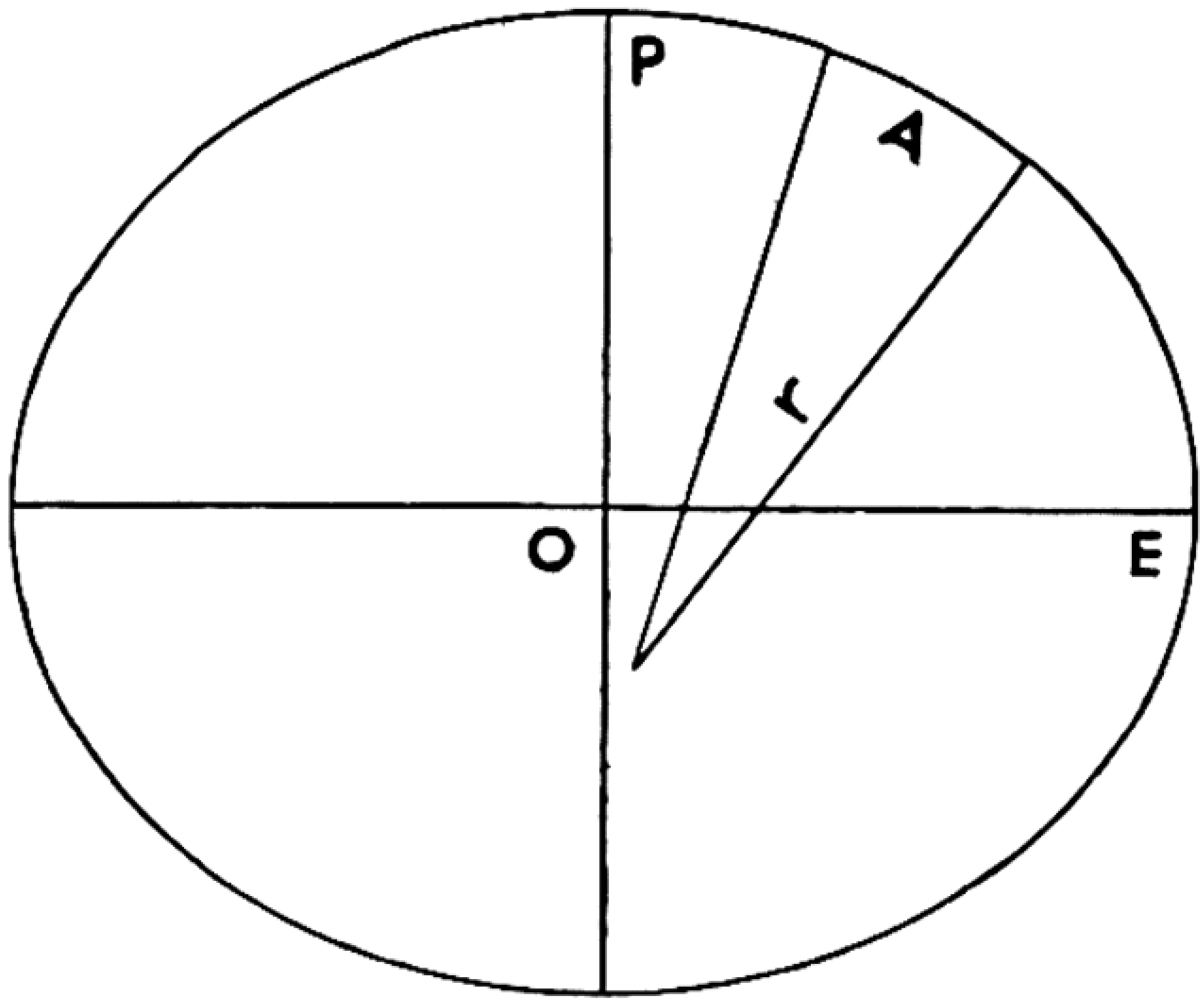


FIG. 19

a meridian is an ellipse (Fig. 19), the following formulae are true.

Put OE = semi-equatorial diameter $= a$

OP = „ polar „ $= b$

and let $\dfrac{a^2 - b^2}{a^2}$ = eccentricity of ellipse $= e$.

This last formula is simply the definition of eccentricity adopted in mathematics.

Also let r = radius of curvature of meridian at any point A, and ϕ = latitude of A.

$$\text{Then } r = \frac{a(1 - e^2)}{(1 - e^2 \sin^2 \phi)^{\frac{3}{2}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now we have seen how (by finding a meridional distance and the latitudes at two stations) we can find the mean radius of curvature between the stations.

Let us take this as giving the exact radius at the point midway (in latitude) between the two stations.

We thus have a radius of curvature along the meridian at one known latitude.

Suppose two such determinations are made, namely r_1 and r_2 in latitudes ϕ_1 and ϕ_2.

Then $\qquad r_1 \, (1 - e^2 \sin^2 \phi_1)^{\frac{3}{2}} = a \, (1 - e^2)$

and $\qquad r_2 \, (1 - e^2 \sin^2 \phi_2)^{\frac{3}{2}} = a \, (1 - e^2)$

so that $\qquad r_1 \, (1 - e^2 \sin^2 \phi_1)^{\frac{3}{2}} = r_2 \, (1 - e^2 \sin^2 \phi_2)^{\frac{3}{2}}$

It will be clear that as all symbols except e in this equation are known, we can find e from it.

Then a can be found from either of the preceding equations.

Knowing a and e, we can find b at once, so that the ellipse is known. In fact, many such arcs must be employed for a good result, in consequence of the unavoidable errors of measurement. This, however, is the principle of the determination of the earth's figure by the measurement of arcs of meridian.

Also if R be the radius of the parallel in latitude ϕ,

$$R = \frac{a \cos \phi}{(1 - e^2 \sin^2 \phi)^{\frac{1}{2}}} \qquad \qquad (2)$$

and, similarly, we can find a and e by finding the distances along two parallels corresponding to given differences of longitude.

If the earth were a sphere of radius a, the above formula would become $R = a \cos \phi$, as e, of course, would be zero.

Mean Sea Level.

It is clear that the irregularities on the earth's surface, such as land surfaces of different elevations, must depart appreciably from any regular shape, such as that of a spheroid. When, therefore, we speak of the figure of the earth, we do not mean the actual surface, but, approximately, the shape which that surface would assume if all the land above sea level were removed. This explains why mean sea-level is chosen as the height to which all base lines should be reduced (as explained in Part I of this work) in large surveys.

Geodetic Surveys.

Surveys intended to supply data for determining the earth's figure are called *geodetic* surveys, in contrast to *topographic* surveys, in which the object is simply the representation of the configuration of the actual surface in one part of the earth.

Nearly all surveys of large areas are now so conducted as to come under the first head.

Gravity Surveys.

In addition to the determinations of lengths of known arcs of meridian and parallel, we can find, as already stated, the lengths of the time of swing of a pendulum in different latitudes. This time varies, in accordance with known mathematical laws, with the force of gravity, after allowing for the effect of the earth's rotation ; and the force of gravity varies, also in a known way, with the distance from the earth's centre and with the latitude. Hence by comparing the times of swing at two places, we can arrive at a conclusion as to how far the shape departs from the spherical.

Now the greater part of the earth's surface is covered by water. Here we cannot measure accurately an arc either of meridian or parallel. But if it be found possible to obtain accurately the time of swing of a pendulum on a ship of any kind, it is clear that this method will enable us to make comparisons on sea as well as on land.

The trouble is that the swing of the pendulum is affected by the motion of the ship.

Within the last few years, however, experiments have been made on submarines with a double pendulum, so arranged that, if hung from fixed supports, the two components would be swinging in opposite directions at the same moment.

Thus any motion of the ship will affect them equally, but in opposite directions, and by a laborious process it is possible to disentangle the results and arrive at a figure for the actual time of swing. Such a method may lead to many new results.

In gravity surveys, the force of gravity at different places may be compared by other methods than taking the swing of an ordinary pendulum, and other instruments have been used for the purpose.

There are also many methods of finding how far the earth's shape departs from the spherical, based on theoretical considerations in mathematics and astronomy.

All these are, however, beyond the scope of this book.

Those who wish to study the matter further are referred to the "Notes on the Geodesy of the British Isles," by Col. C. F. Close, being No. 3 of the Professional Papers, New

Series, issued by the Ordnance Survey. It contains a full bibliography.

Results.

The figures in the table of results given below have been taken, in part, from that paper, and some additional columns inserted.

Name	Date	Equatorial Radius	$\frac{1}{\text{Compression}}$ [1]	Polar Radius	Mean Radius
Airy . . .	1830	6,377,542	299·93	6,356,237	6,366,889
Everest . .	1830	6,377,304	300·80	6,356,103	6,366,703
Bessel . .	1841	6,377,397	299·15	6,356,078	6,366,738
Clarke . .	1858	6,378,293	294·26	6,356,616	6,367,455
Clarke . .	1866	6,378,206	294·98	6,356,582	6,367,394
Clarke . .	1880	6,378,249	293·46	6,356,510	6,367,381
Helmert . .	1906	6,378,200	298·30	6,356,817	6,367,509
Hayford . .	1911	6,378,388	296·96	6,357,489	6,367,938
General Mean .		6,377,947		6,356,555	6,367,251

It is remarkable that these figures seem to show, on the whole, a distinct *upward* tendency, which may be significant, but is more likely to be accidental.

To show how far it is marked, we have taken, in each case, the difference between the general mean radius and each separate determination.

The results are plotted in Fig. 20, where a *minus* quantity indicates a result below the general mean, and a plus result is higher than the mean. It will be seen that the early results are *minus*, the later *plus*. Of course, in some cases the computers have made use of surveys carried out at earlier dates as well as those which were recent, so that the results are not completely chronological. All results are in *international metres*. Col. Clarke gives the relation of this unit to the British yard as 1 yd. = ·9143992 metre.

[1] The term *compression* (col. 4) stands for the value of $\frac{a-b}{a}$, where a and b are the equatorial and polar radii respectively.

It will be clear from the figures that the polar and equatorial radii given by Hayford's calculation are both noticeably greater than those of any previous determination. They are derived from the survey of North America (United States)

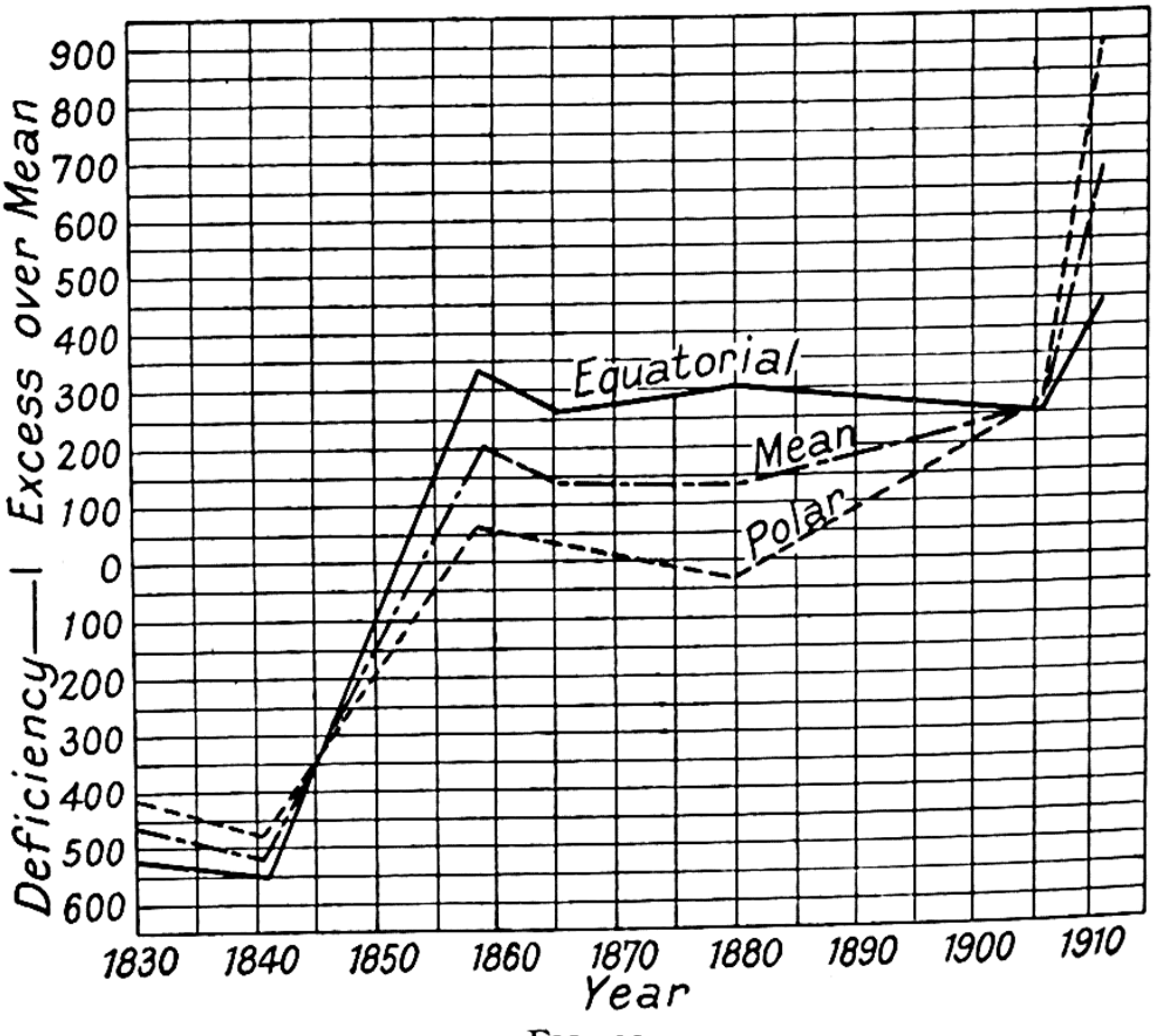


FIG. 20

only, and may either indicate increasing dimensions of the earth, or a greater radius of curvature at that particular part of the surface than elsewhere. These values were, however, adopted for general recommendation at a conference of the International Geodetic Association in 1924, called together for the purpose of deciding which figure should be recommended.

It is only for the most exact work that it makes much difference *which* figure is chosen. The tables in Chapter IX give lengths of 1° of meridian and parallel in different latitudes.

The Theodolite.

We shall conclude this section with a diagram of a theodolite

(Fig. 22), a table of references to show the names of the parts —(p. 53)— and some further notes about it.

In Part I of this work (p. 78) we have given a list of errors to which a single observation of an angle is liable. Some of these will now be referred to more fully.

(*a*) Errors due to the vertical axis not being exactly coincident with the centre of the graduated circle. We have stated that these are eliminated by reading opposite verniers.

In Fig. 21, let B be the centre of the vertical axis, and A the centre of the graduated circle, and suppose that in measuring an angle the vernier moves from X to X_1. It turns round the centre B, so the angle turned through is XBX_1. But the difference between the readings will give the angle at the centre of the circle, namely XAX_1. Let AX_1 meet XB in C.

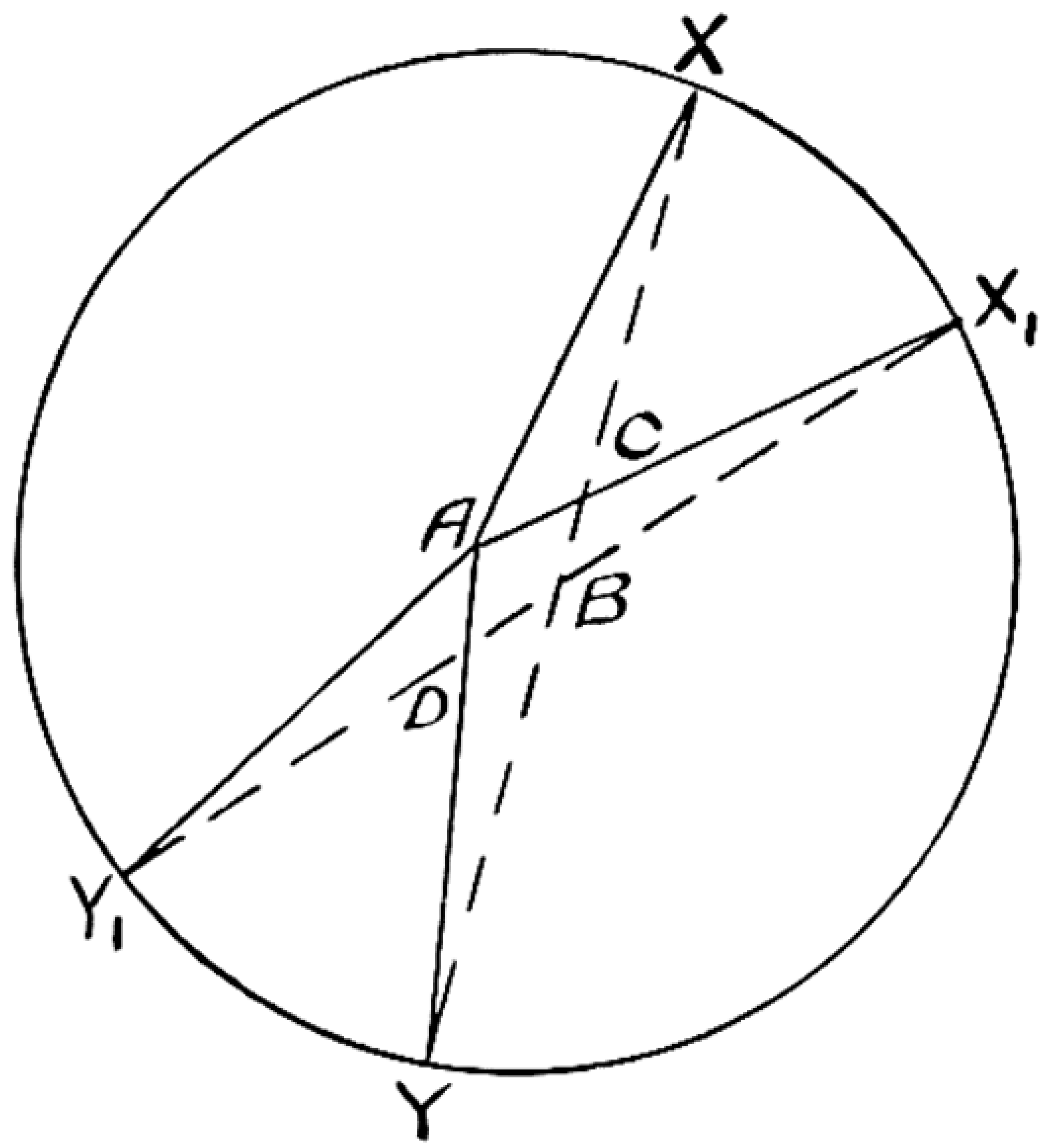


FIG. 21

Now $\quad XCX_1 = XBX_1 + BX_1C$

and $\quad XCX_1 = XAX_1 + CXA.$

$$\therefore\ XBX_1 + BX_1C = XAX_1 + CXA \quad . \quad . \quad . \quad (1)$$

Now suppose there is another vernier Y, opposite to X, so that while X moves to X_1, Y moves to Y_1, also, of course, round the same centre B and through the same true angle $YBY_1 = XBX_1$.

The difference between the readings at Y and Y_1 will give the angle YAY_1.

Let $\quad AY$ meet BY_1 at D.

Then $\quad YDY_1 = YBY_1 + DYB$

and $\quad YDY_1 = YAY_1 + AY_1D.$

$$\therefore\ YBY_1 + DYB = YAY_1 + AY_1D \quad . \quad . \quad . \quad (2)$$

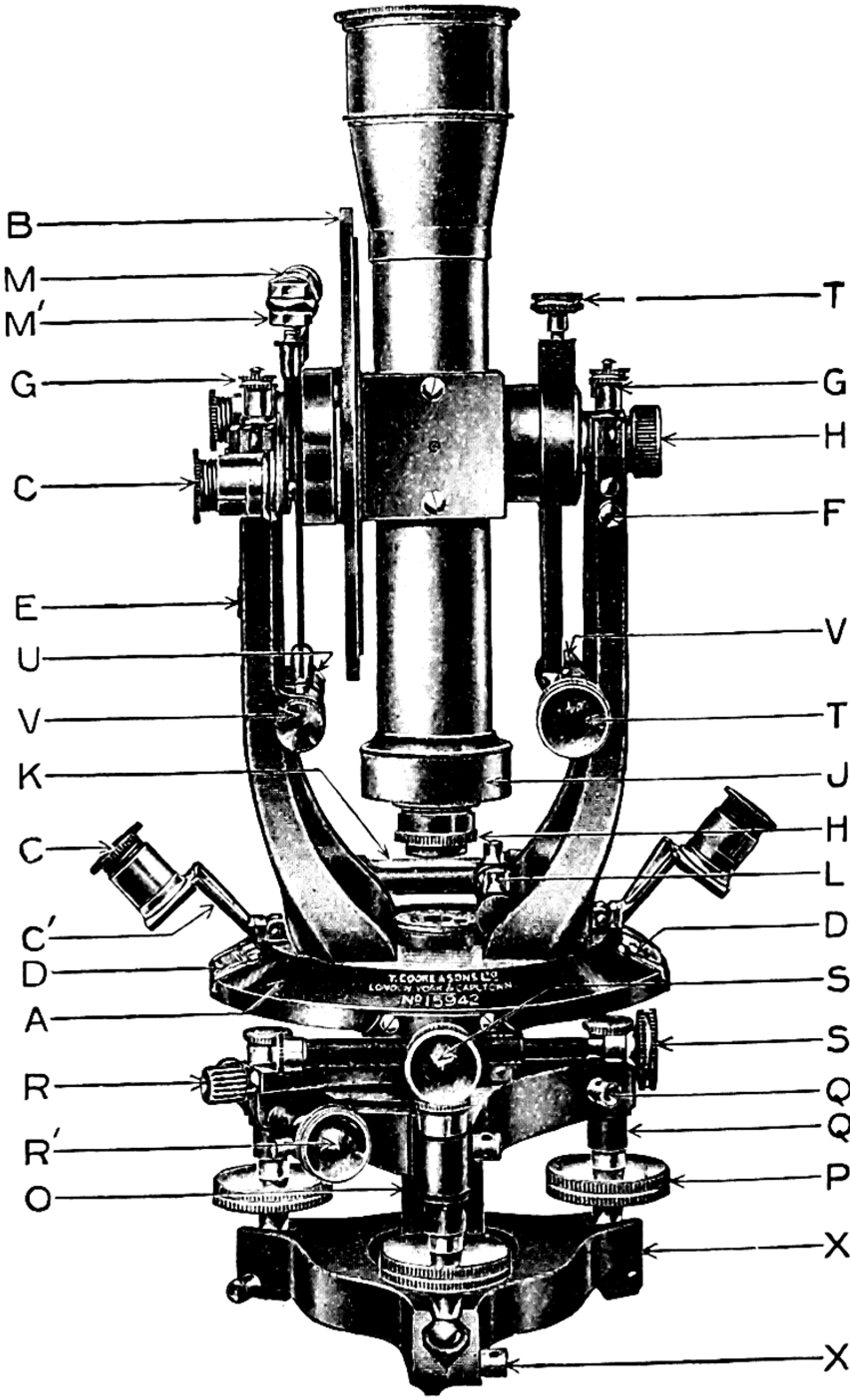


FIG. 22

(For Key see foot of page 53)

Now add (1) and (2) together.

$\therefore XBX_1 + YBY_1 + BX_1C + DYB = XAX_1 + YAY_1 + CXA + AY_1D.$

Now $AX_1 = AY_1$ as A is the centre of the circle.

$\therefore BX_1C = AY_1D$;

similarly $DYB = CXA$.

$$\therefore BX_1C + DYB = CXA + AY_1D.$$

$$\therefore XBX_1 + YBY_1 = XAX_1 + YAY_1.$$

Now XBX_1 and YBY_1 are equal, and either of them gives the true angle, while XAX_1 and YAY_1 are the measured values of the angle as given by the opposite verniers.

Hence we have $2 \times$ true angle = sum of measured values, or the true angle is the mean of the measured values.

(*b*) Errors due to faulty adjustment of the theodolite.

It is proposed to refer to three of these.

In Fig. 23, let AB be the horizontal or trunnion axis of the telescope, along the line of the arrow H^1, in Fig. 22.

Then, (*a*) when the instrument is levelled, this axis should be truly horizontal.

Next let K be the centre of the cross hairs, and O the optical centre of the object glass.

Then KO is called the "line of collimation," and is the line along which we actually read.

KEY TO FIG. 22

A. Horizontal circle, bevelled $22\frac{1}{2}°$.
B. Vertical circle, flat.
C. Readers for circles.
C'. Reflectors for verniers.
D. Removable glass covers to verniers.
E. Clamping face for magnetic compass.
F. Antagonistic screws for adjusting height of transit axis.
G. Clamp screw for caps of transit axis pivots.
H. Screw-focusing eyepiece.
H'. Focusing head for telescope.
J. Diaphragm adjusting screws at *sides* of telescope (not seen).
J'. Diaphragm adjusting screws at *top* and *bottom* of telescope (not seen).
J". Protecting cap and dust excluder for *J* and *J'*.
K, *L*. Plate bubbles and nuts for adjusting same.
M. Vertical vernier arm bubble (altitude zero setting).
M'. Nuts for adjusting *M*.
O. Tribrach.
P. Footscrews.
Q. Dustcovers for footscrews.
Q'. Tightening screws for footscrews.
R. Clamp for lower plate.
R'. Slow-motion screw for lower plate.
S. Clamp for upper plate.
S'. Slow-motion screw for upper plate.
T. Clamp for telescope.
T'. Slow-motion screw for telescope.
U. Slow-motion screw for vertical vernier bubble (*M*).
U'. Check-nut for *U*.
V. Quick release spring boxes for vertical slow-motion screws *T'* and *U*.
X. Trivet stage.
X'. Screws for tightening up ball-ends of footscrews.

(*b*) This line *KO* should be truly perpendicular to the trunnion axis *AB*.

If both these adjustments are perfect, it is clear that as we turn the telescope round *AB*, the line *KO* will describe a true vertical plane, because *AB* will be horizontal and *KO* perpendicular to it.

Suppose, however, that *KO* is perpendicular to *AB*, but *AB* is not horizontal (Fig. 24). Then as we turn the telescope round *AB*, the line of sight *KO* will describe an *inclined* plane, instead of a vertical one. In Fig. 24, *HH*, *VV* represent the true horizontal and vertical lines, and in the case shown suppose we look along the line *KO* at a point high up (say a triangulation station on top of a hill), and that the next point we wish to observe is much lower down, and to the right of the first.

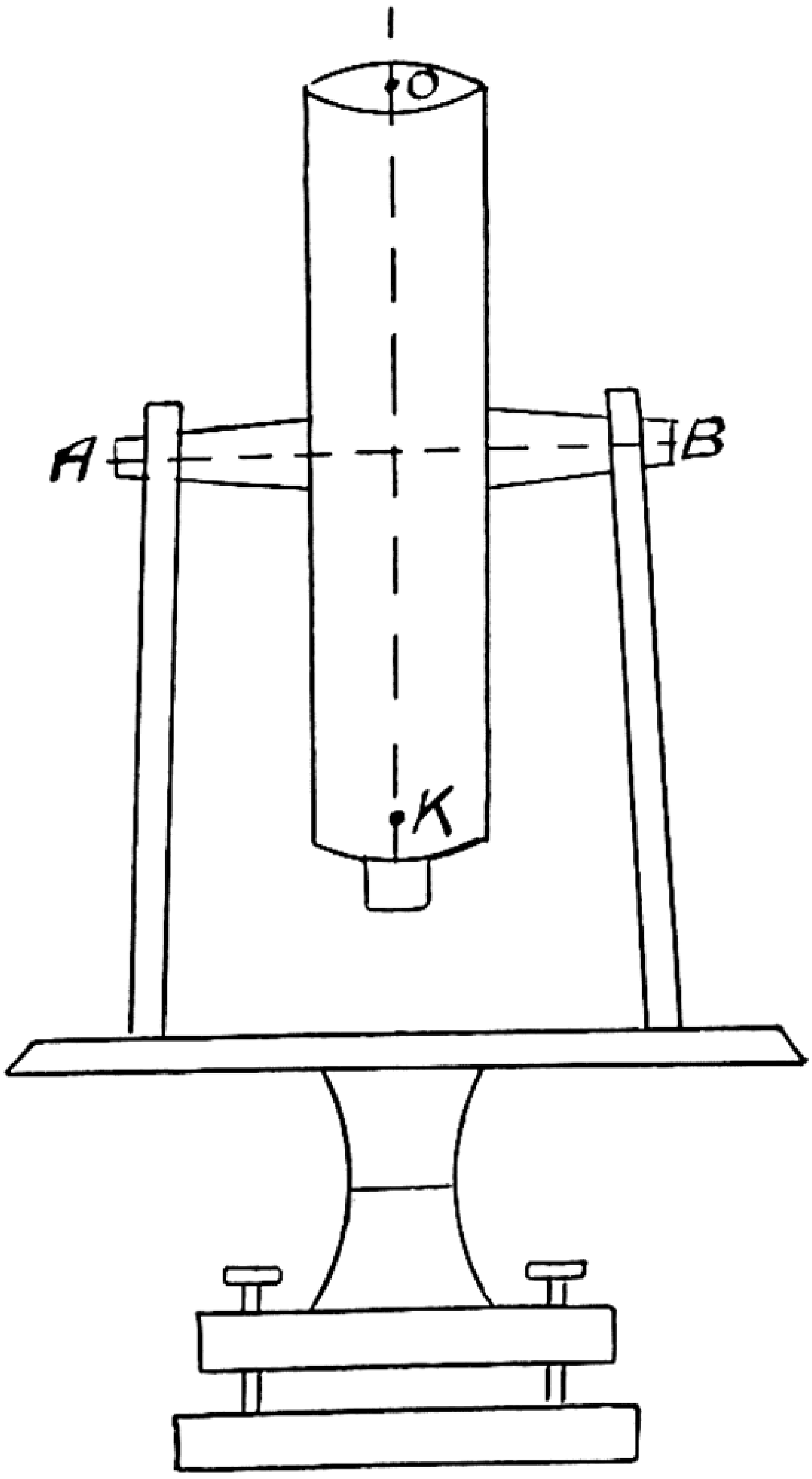


FIG. 23

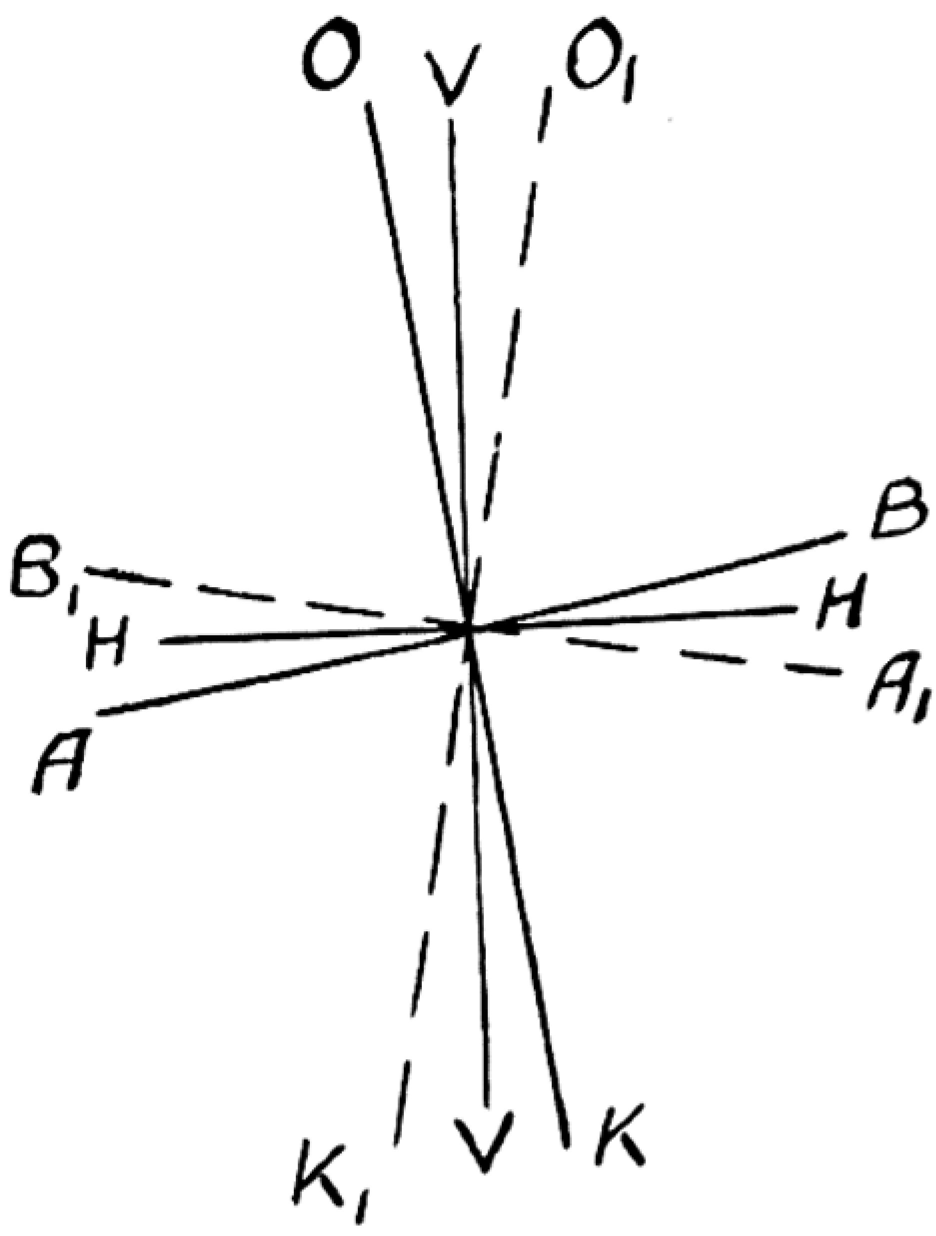


FIG. 24

In turning the telescope *down* to the level of the second station, it is clear that the line of sight, instead of coming down vertically, will move to the right a little, before we start

turning the horizontal plate at all. Hence the observed angle will be *too small*.

If now we reverse face, this turns the theodolite round. The high side, B_1, of the axis will now be on our left, as shown by the dotted line B_1A_1, and if we look again at the high station and turn the telescope down once more (after taking the reading) then the line of sight, K_1O_1, will move this time to the *left*, in coming down, by the same amount as it moved to the right before. Hence when we turn the theodolite to the *right* to look at the second station, we must turn through an angle which is too *big*, by the same amount as it was too small before.

Hence the *mean* of the face-left and face-right readings will give the true value.

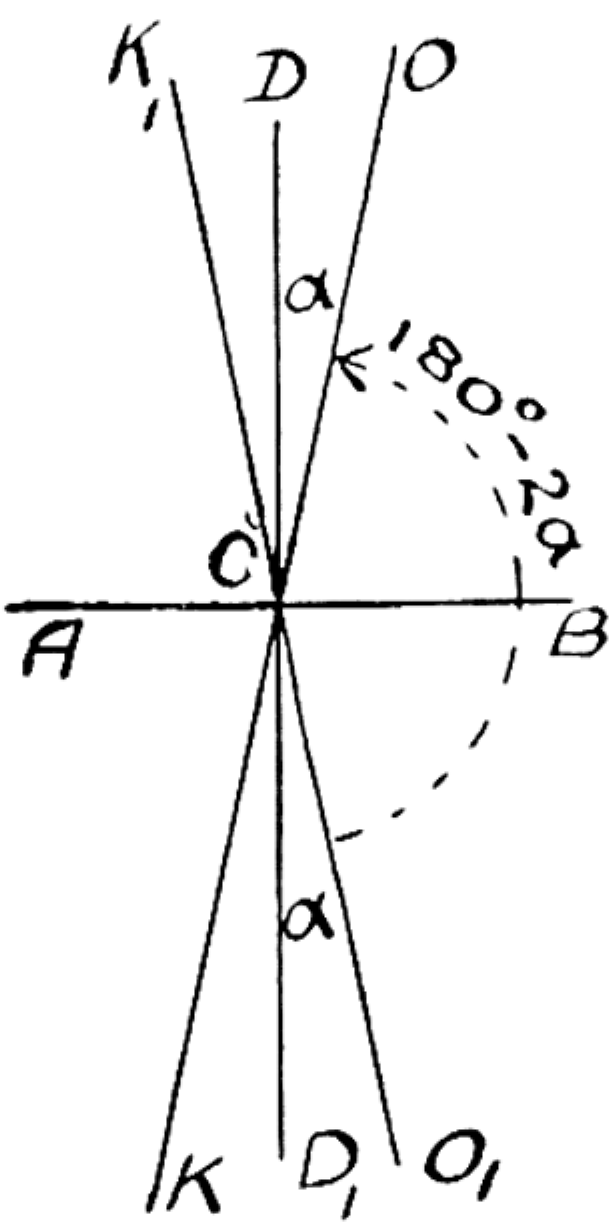


FIG. 25

Next suppose that AB is truly horizontal, but KO is not perpendicular to it.

In Fig. 25, the reader is asked to imagine that AB is the horizontal axis of the theodolite, D_1D is a perpendicular to it, and KO is the line of collimation, or line of sight. Also that the telescope is horizontal, and that he is looking *down* on all these lines, as seen in *plan*, and let the angle $DCO = \alpha$, which represents the faulty adjustment of KO. (C is the centre point.)

Now suppose we turn the telescope over *vertically*, round the axis AB. Then it is clear that the line KO will describe a double cone with its apex at C, and if we go on until the telescope is horizontal again (but turned over), then the line KO will have moved to K_1O_1.

This process of turning the telescope over is called *transiting* the telescope. It can only be done with a *transit* theodolite, which explains the reason for the name.

If the adjustment were perfect, the line of sight would be D_1D. After transiting, it would be DD_1, and the angle DCD_1 would be exactly 180°, so that we should be looking in diametrically opposite directions.

If we set on a mark at D before transiting, and read the

verniers, then transit, and turn back *horizontally* to D, the readings should differ by exactly 180°.

But if the adjustment is faulty, as we have seen, the line of sight moves from CO to CO_1, and the angle OCO_1 is $180° - 2\alpha$. Hence if we set on a mark O, read, transit, and turn back horizontally to O, the new readings will *not* differ from the first by 180°, but by $180° \pm 2\alpha$, according to the direction of the error.

If we take readings with the telescope *not* horizontal, the amount of discrepancy, due to the same error α, will be increased. Thus the errors in the readings to two points at different altitudes would not be the same, and would affect the value of the angle between them.

But it is clear that if we reverse face the telescope is turned over and the *direction* of the error is reversed for each point, so that, as before, the mean of the readings (face-left and face-right) gives the true value.

(*c*) For reading vertical angles, when the instrument is carefully levelled, and the vernier of the vertical circle is set to read zero, the line of collimation should be truly horizontal.

If not, it will be pointing up or down at an angle called the *index error* of the vertical circle. If it is pointing *upwards*, then it is clear that by turning the telescope over, and re-setting the vernier to zero, we cause the line to point *downwards* at the same angle. Hence if we read an angle of elevation face-left and face-right, the mean of the results will give the true value.

We may also find the amount of the index error as on page 13, and apply it as a correction to all vertical angles.

CHAPTER III

SIMPLE TRIGONOMETRICAL CALCULATIONS

Traverse Co-ordinates.

In dealing with traverse surveys, by compass, in Vol. I of this book, we have shown that if the bearing NAB of a line AB (Fig. 26) is known, and also the angle ABC between that line and the next line BC (measured clockwise from the back station as shown by the full arrow), then the bearing of the next line can be found by a simple rule, and we have shown

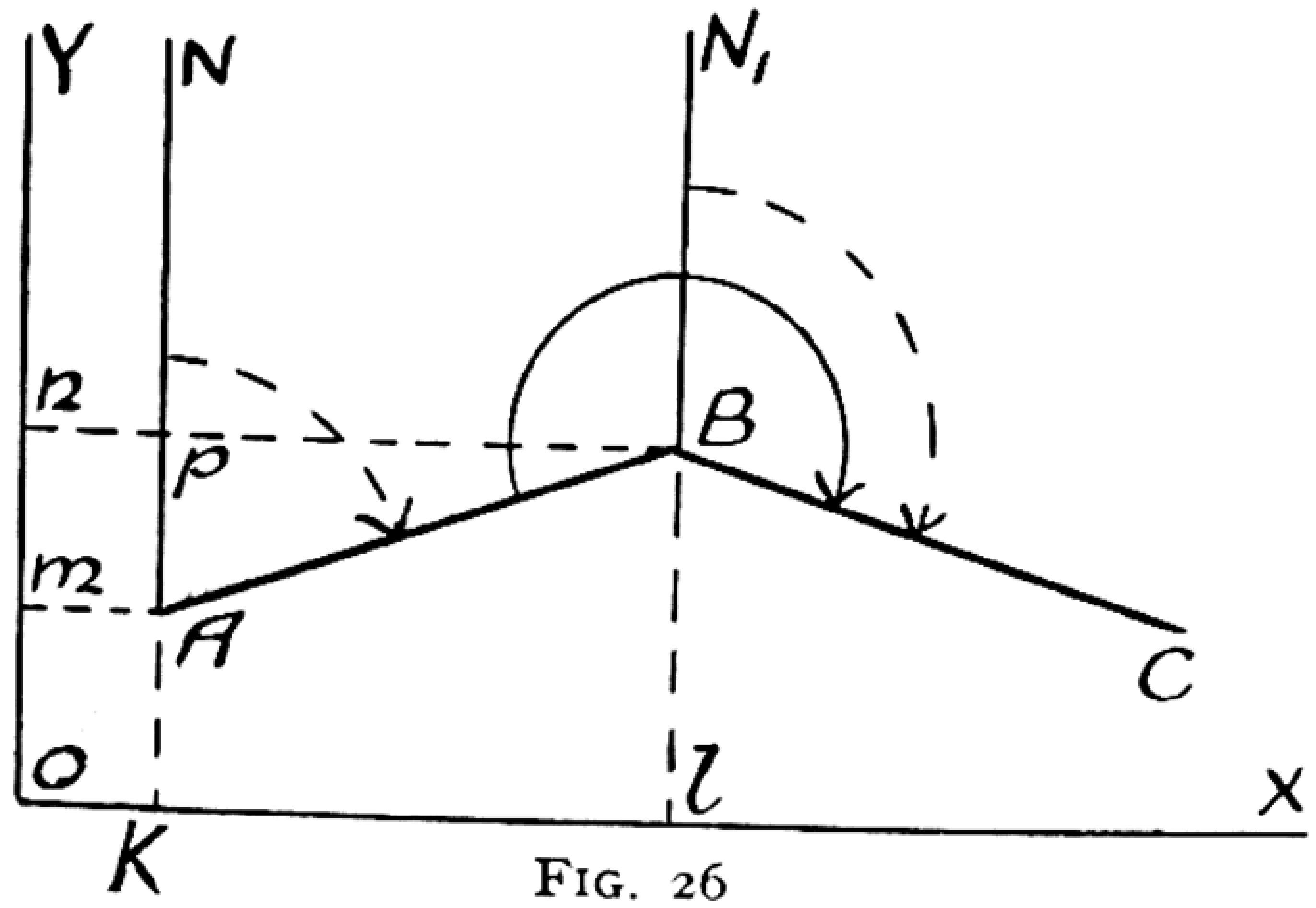


Fig. 26

how these bearings can be used for plotting by scale and protractor.

In theodolite traverses, the included angles are read by theodolite with much greater accuracy than is possible with the compass. Assuming that the distances are also measured with a corresponding degree of accuracy, the errors in the field work are likely to be much smaller than the unavoidable errors of drawing which would accrue in any system of plotting by scale and protractor.

Hence it is usual to assume two axes of co-ordinates, OX, OY, and to *calculate* the distance of each station from each of these axes.

These distances are called the *co-ordinates* of the station.

Thus kA and mA are the co-ordinates of A, while lB and nB are those of B. When these are known, each station can be plotted independently, so that there is no accumulation of plotting errors.

In calculating co-ordinates, the steps are as follows: (1) The bearing of the first line must be known. This may be measured by compass if magnetic north is to be at the top of the paper. If geographic north is to be at the top, the bearing must be a true or geographic bearing. This may be found either by correcting the magnetic bearing for declination (Vol. I, p. 27), or by a direct observation for azimuth (this volume, p. 31).

But if the exact direction of north is not a matter of much importance (the object being to plot the correct shape of the traverse simply), then any bearing may be *assumed* for the first line which the surveyor thinks will make the drawing fit well on the paper.

(2) The included angles must be known. These must be measured clockwise from the back station. The methods of measuring them by theodolite is given on page 77, Vol. I.

(3) The bearings are found by adding each included angle to the previous bearing; then, if the result is *less* than 180°, *add* 180°; but if *more* than 180°, subtract 180°. This will give the bearing of the next line, and so on. If the traverse is closed, the angles should be summed, and corrected if necessary, before calculating bearings (Vol. I, p. 35).

The bearings so found are *whole-circle*, and each may go up to 360°. They are all clockwise from the top of the paper.

We shall henceforth speak of the top of the paper simply as "north," though it may be true north, magnetic north, or only an assumed north, as above.

(4) The *reduced bearings* are calculated.

Reduced bearings are measured either from *north* or *south* (whichever may be the nearer), and always the shorter way, so that every reduced bearing is under 90°. They are *never* measured from east or west.

In Fig. 27 the whole circle bearings of OA, OB, OC, OD are marked with full arrows, the reduced bearings dotted.

Thus for the line OB the whole circle bearing is the clockwise angle NOB. The reduced bearing is the angle SOB.

The whole circle bearing of OS is 180°; that of ON may be either zero or 360°. Hence all reduced bearings are measured

from zero, from 180°, or from 360°, according to which of these is nearest to the whole circle bearing. In the case of OB this line is nearest to 180°, and the reduced bearing is $180° - NOB$.

In order to tell in which quadrant the line lies, we must now state the cardinal points as well as the reduced bearing: East is 90°, south 180°, and west 270°.

Hence if a whole circle bearing lies between 90° and 180°

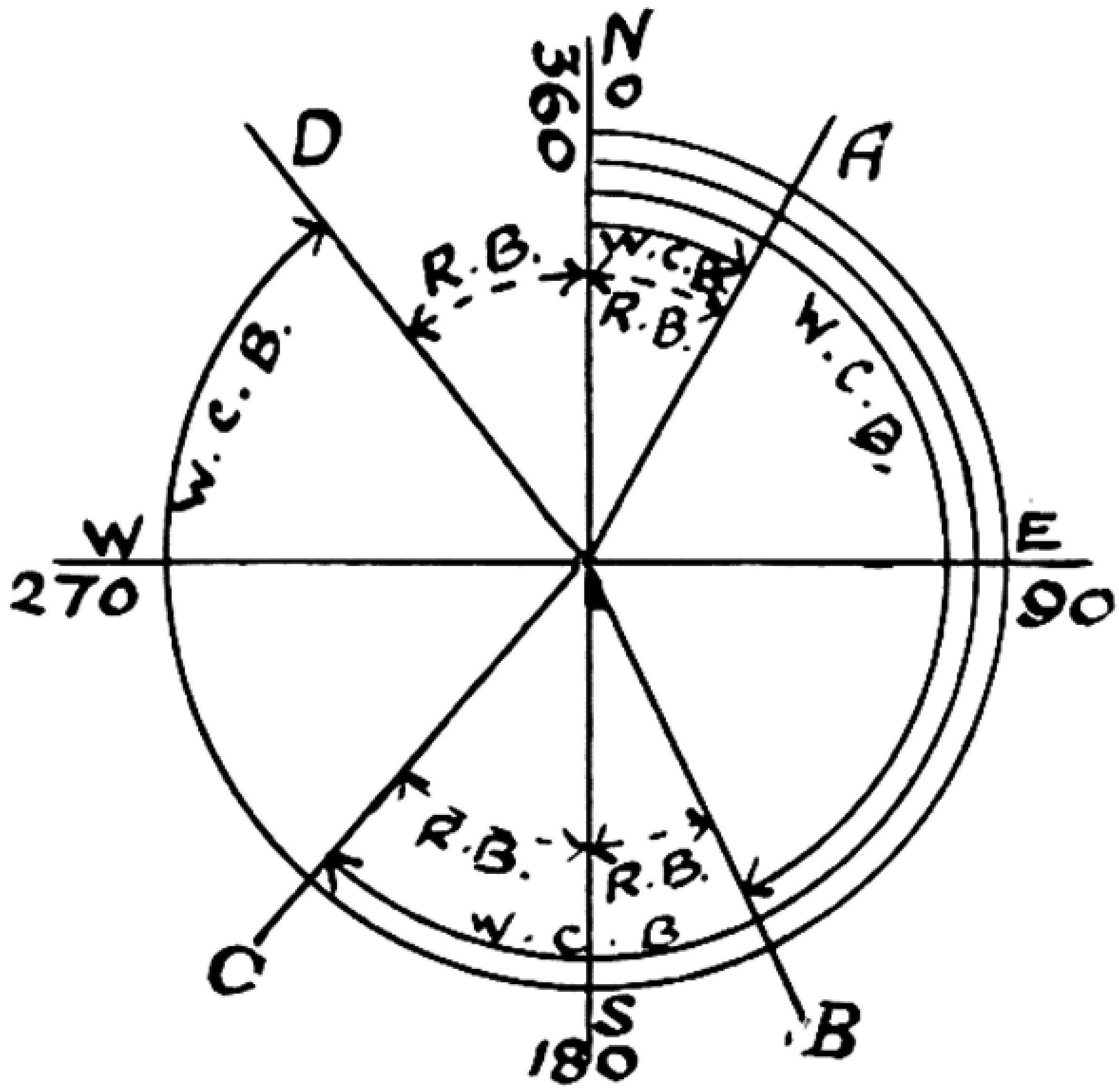


FIG. 27

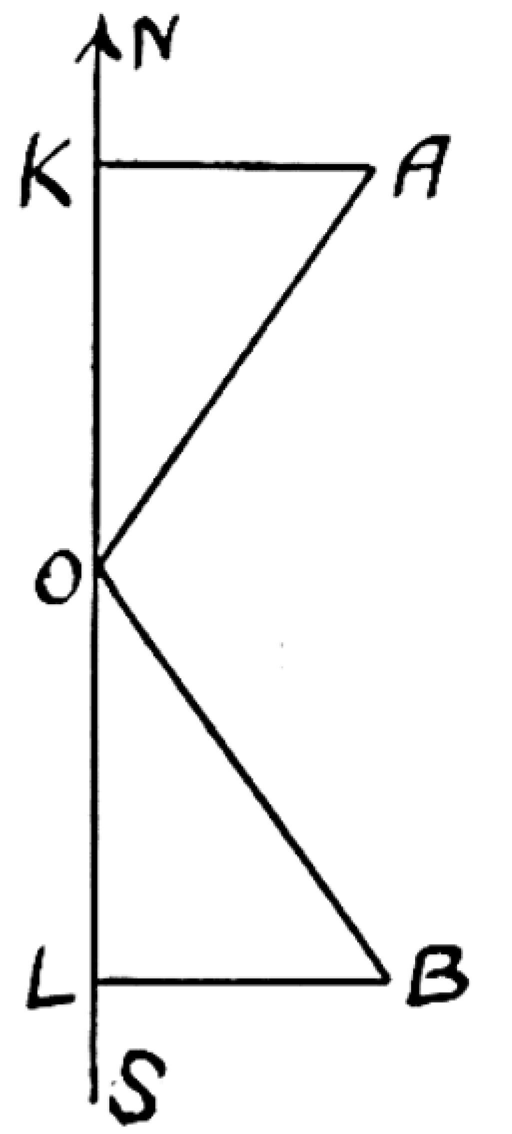


FIG. 28

it is between *east* and *south*. Its cardinal points are said to be *south-east*, and so on.

Thus, suppose the whole circle bearing of OC is 242° 54′. This is between south (180°) and west (270°). Hence it is nearer to south (180°) than to north (360°). The cardinal points are south-west, and the reduced bearing is measured from 180°.

Hence, reduced bearing = 242° 54′ − 180° = 62° 54′ S.W.

Some people prefer to write this as S. 62° 54′ W., the letters S.W. being understood sometimes to mean exactly midway between south and west, whereas here they mean *anywhere* between the two.

If the whole circle bearing were 342° 54′, the reduced bearing would be 360° − 342° 54′ = 17° 6′, and the cardinal points N.W.

(5) *Differences of latitude* and *departures* are calculated.

If OA (Fig. 28) be any line, it is clear that in moving along

OA, we shall travel a certain distance, OK, in the north (or south) direction. This is called the *difference of latitude* from O to A for the purpose of traverse calculations.

We shall also travel a certain distance, KA, *eastwards* (or westwards, as the case may be), and this is called the *departure.*

For the line OB, OL is the difference of latitude, and LB is the departure.

Now the angles KOA and LOB are the reduced bearings of these lines, and OA, OB are their lengths.

Hence clearly we have the following rules—

Difference of latitude = length of line
× cosine of reduced bearing,
and departure = length of line × sine of reduced bearing.

The student must learn these.

Thus $OL = OB \cos LOB$; $LB = OB \sin LOB$, and so on.

The difference of latitude is north or south according to the *first* cardinal point, the departure east or west according to the *second.* Thus, if the whole circle bearing of a line is 137° 27′, and its length is 675·8 ft., we have the following results—

Reduced bearing = 180° − 137° 27′ = S. 42° 33′ E., or 42° 33′ S.E.

log. cos 42° 33′	= 9·86728	log. sin	= 9·83010
log. 675·8	= 2·82982		2·82982
Log. diff. of lat.	= 2·69710	log. dep.	= 2·65992
diff. of lat.	= 404·7 S.	dep.	= 457·0 E.

If the traverse is closed, the results are summed. As we come back to the starting point, it is clear that all the *north* differences of latitude must add up to the same total as those which are *south* ; and the sum of the *east* departures must be the same as the sum of the *westings.*

This checks not only the calculations but also the accuracy of the chaining of the lines. If there is a big discrepancy this shows a mistake somewhere.

But *small* closing errors (which should not exceed 1 part in 2,000 for only moderately good work, or about 1 part in 20,000 for really good work with special precautions to ensure accuracy) are put down to small chaining errors, and are

distributed over the different lines, so as to make the figures close exactly.

(6) *Co-ordinates* are now assumed for the first point. These are guessed, so as to get the drawing suitably placed on the paper. Starting from this point, to the *north* co-ordinate we add the difference of latitude to the next, if it is *north* (or

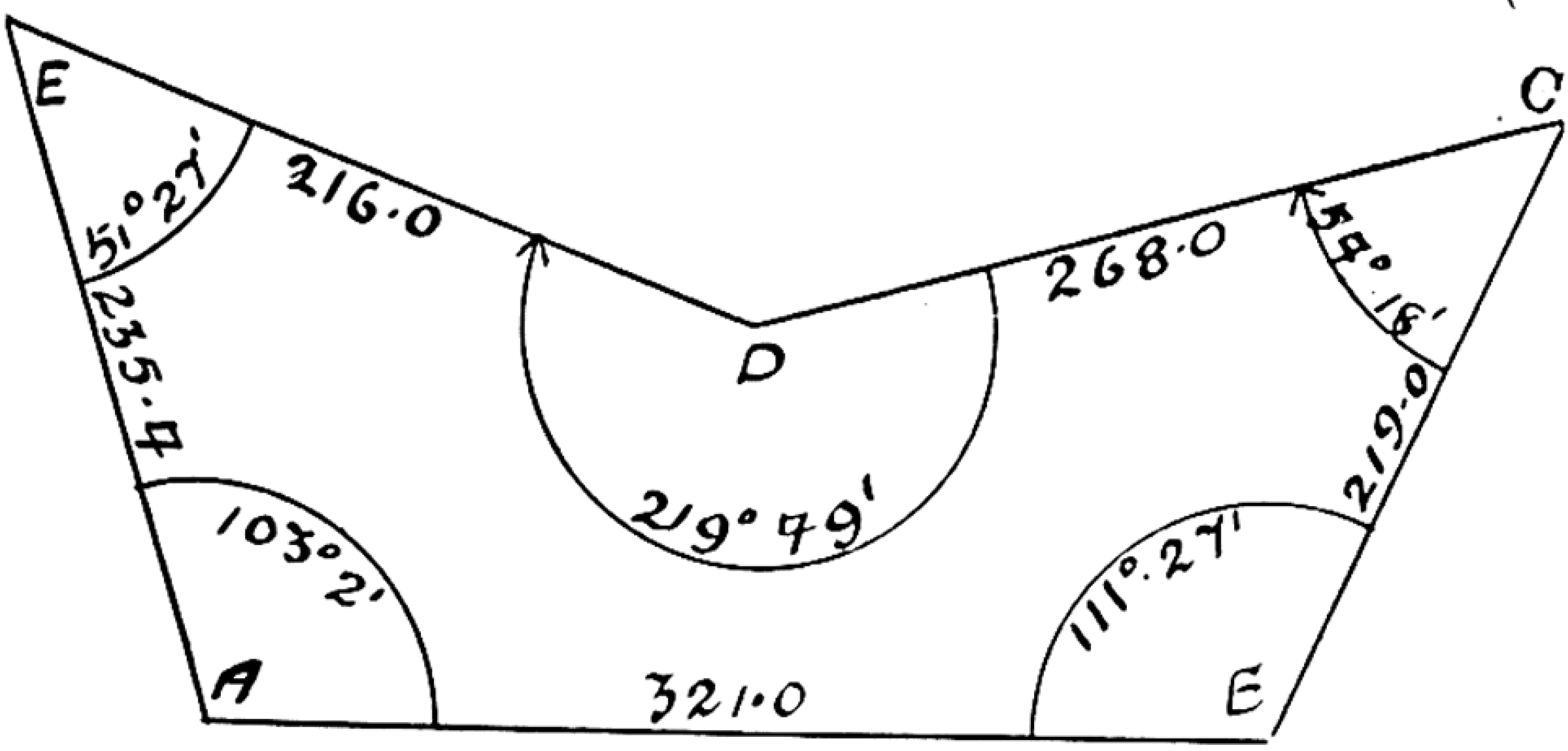


FIG. 29

subtract it, if south), and the result will give the north co-ordinate of the next point.

Similarly the departure is added to the east co-ordinate if east, or subtracted if west, thus giving the next *east* co-ordinate; and so on.

Thus going back to Fig. 26, page 57, the north and east co-ordinates of A are kA and mA respectively. The difference of latitude and departure, A to B, are Ap north, and pB east.

The co-ordinates of B are lB and nB, and it is clear that

$$lB = kp = kA + Ap, \text{ and}$$

$$nB = mA + pB.$$

Example.

Take the case shown in Fig. 29. Suppose the angles and distances marked have been observed, and that it is desired to plot the survey with AB horizontally at the bottom of the drawing.

Checking the angles: the sum of the observed angles is 540° 3′. It should be 540°. Assuming that we need only work to minutes, there is one minute to be subtracted from

each of three angles, and we choose those adjacent to the shortest sides, namely, *C*, *D*, and *E*, so that those angles become 54° 17′, 219° 48′, and 51° 26′.

The bearing of *AB* must be taken as 90°, because it is to be horizontal, and therefore at 90° to the top of the paper. The remaining bearings are calculated as described above and illustrated in Vol. I, and the differences of latitude and departure are found from the rules diff. of lat. = length × cosine reduced bearing, and departure = length × sine reduced bearing.

Line	Whole Circle Bearing	Reduced Bearing	Cardinal Points	Length	Difference of Latitude				Departure			
					N.	Corr.	S.	Corr.	E.	Corr.	W.	Corr.
	° ′	° ′										
AB	90 0	90 0	E.	321·0	—	—	—	+ ·4	321·0	− ·1	—	—
BC	21 27	21 27	N.E.	219·0	203·8	− ·3	—	—	80·1	—	—	—
CD	255 44	75 44	S.W.	268·0	—	—	66·0	+ ·3	—	—	259·7	+ ·1
DE	295 32	64 28	N.W.	216·0	93·1	− ·3	—	—	—	—	194·9	—
EA	166 58	13 2	S.E.	235·4	—	—	229·3	+ ·3	53·7	—	—	—
				1259·4	296·9	—	295·3	—	454·8	—	454·6	—

The correction required, in latitude, to make the northings and southings balance is 296·9 − 295·3 = 1·6 ft. Perhaps the best method of distributing it is by Bowditch's method, already referred to (in its graphical form) in Vol. I. Put into mathematical form, it becomes—

Correction to any diff. of lat.

$$= \text{total error in lat.} \times \frac{\text{length of side considered}}{\text{sum of all the sides}}$$

Thus the correction to the first lat. diff. is $1{\cdot}6 \times \dfrac{321{\cdot}0}{1259{\cdot}4}$, and so on. This is only required to one decimal place, and may be guessed nearly enough. The results are *added* to *southings* (in this case, as southings are too small), and subtracted from northings. They are entered, with the proper signs, in the correction columns.

Finally, co-ordinates are chosen for *A*, say 100 N. and 100 E., so as just to leave room for any detail at the bottom and left of the sheet. The calculation of the remaining

co-ordinates is shown below, using the corrected differences of latitude and departures.

		N.		E.
A		100		100
AB	−	0·4	+	320·9
B		99·6		420·9
BC	+	203·5	+	80·1
C		303·1		501·0
CD	−	66·3	−	259·8
D		236·8		241·2
DE	+	92·8	−	194·9
E		329·6		46·3
EA	−	229·6	+	53·7
A		100·0		100·0, which checks.

The paper can then be ruled off in squares, each representing, say, 100 ft., numbered both ways from zero in the bottom left-hand corner, and the points plotted from the co-ordinates.

The student should work through the whole example.

The drawing is checked by measuring the lengths of the lines.

Traverse Based on Triangulation.

Where a traverse joins two triangulation stations, instead of being an independent survey, certain modifications are necessary. Thus if X (Fig. 30) be the starting (triangulation) station, we should read the included angle YXA, between a triangulation line XY and the first traverse line, XA, to start with. Then in calculating bearings we must start with the known bearing of YX (*not* XY) from the triangulation, and calculate the rest of the bearings as above.

To check the angles, when we arrive at the terminal triangulation station Z, we observe the angle BZW between the last traverse line BZ and a triangulation line ZW.

Hence we can find the bearing of ZW as given from the traverse. This must agree with its known bearing from the

triangulation, and if it does not do so exactly, any small error is distributed over the different angles to make the result come right.

Then we must not *assume* any co-ordinates. The co-ordinates of X will be known from the triangulation survey, and we must start with those. The northings and southings will no longer be equal, but the sum of the northings from X to Z *minus* the sum of the southings, must clearly agree with the result obtained by subtracting the north co-ordinate of X from that of Z. Similarly the difference between total eastings and westings must be equal to the difference between the east co-ordinates of X and Z.

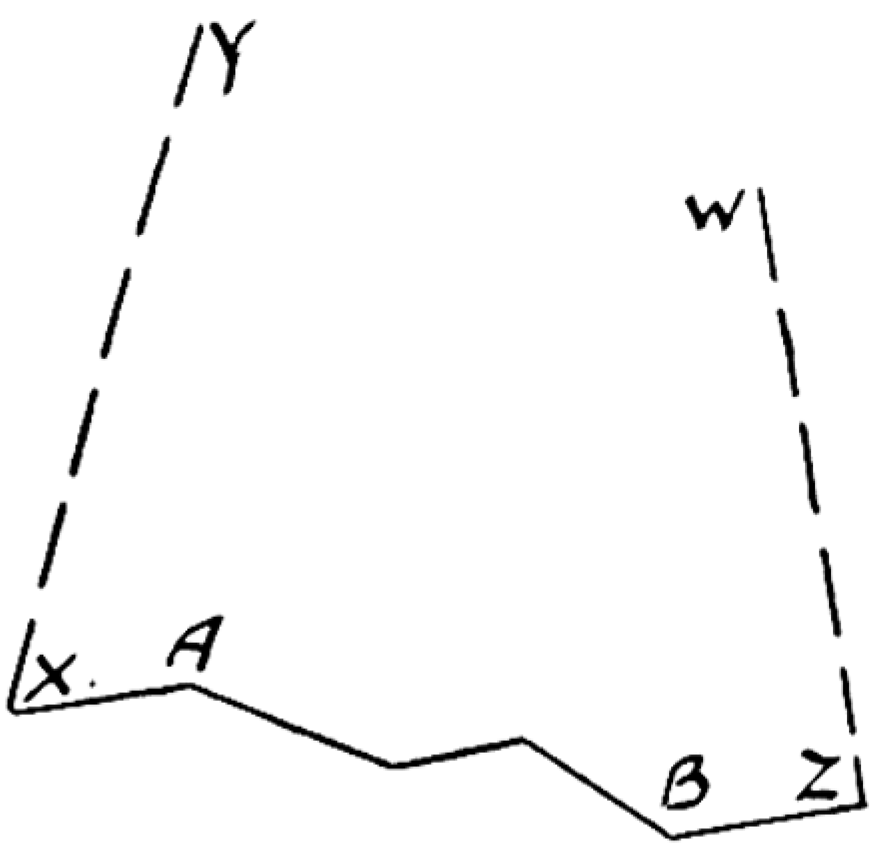


FIG. 30

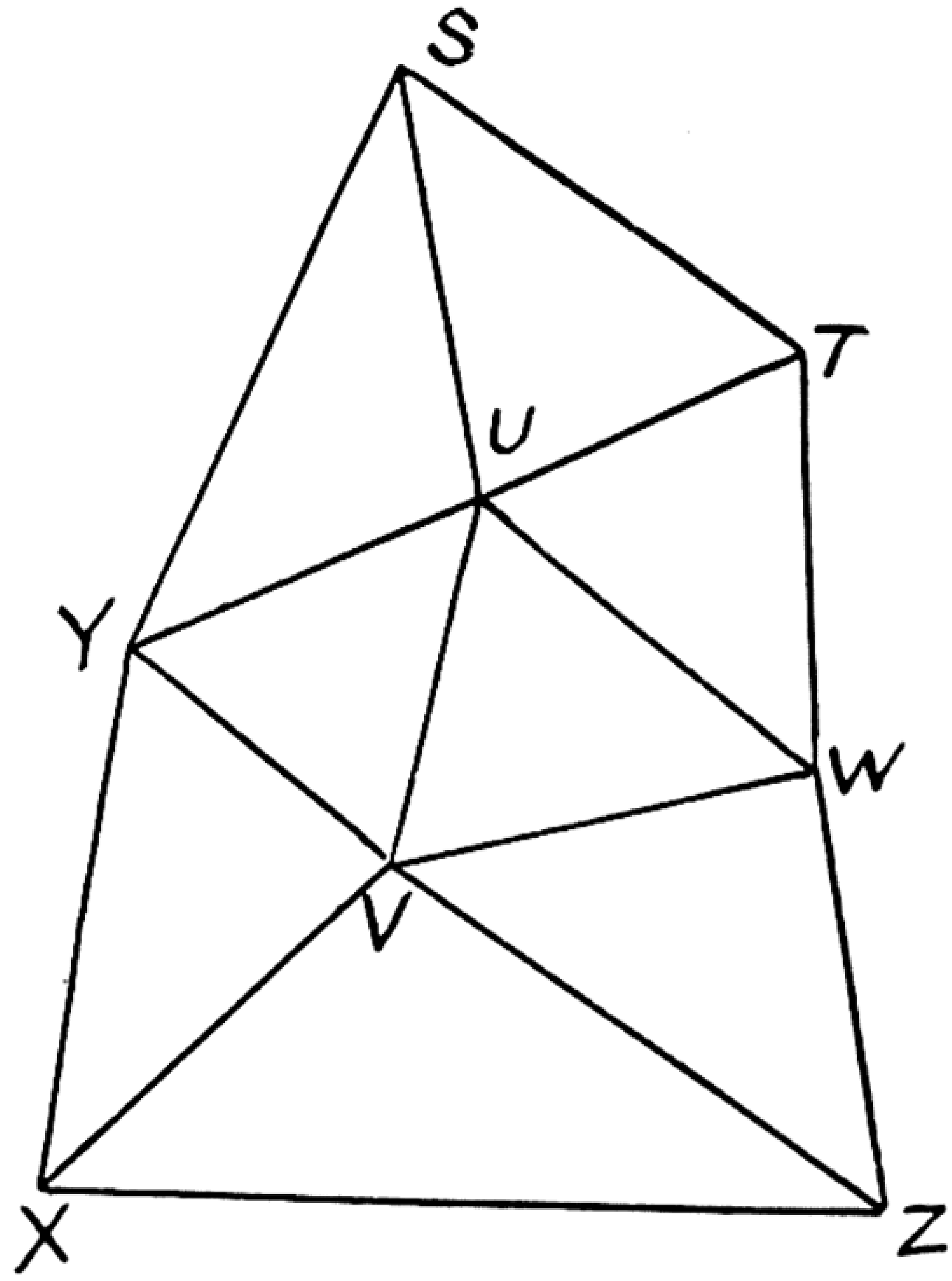


FIG. 31

Small errors are distributed as before, and intermediate co-ordinates can then be found.

Calculating Co-ordinates for a Triangulation.

We have stated in Vol. I the method of measuring the angles in a small triangulation survey. Bigger ones are measured on the same principle, but with more care. We have also stated that the measured angles are subjected to certain tests, and adjusted mathematically to make them satisfy these tests.

Then a base line (say YU, Fig. 31) is very accurately

measured, and all the remaining sides are calculated from this, using the corrected angles.

For small triangulation, with sides not exceeding a mile or so, this may be done by the usual formulae for plane triangles, namely—

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

where a, b, c are the sides of the triangle, and A, B, C, the angles opposite to them. The calculation is done systematically. In the case shown, we should start from the base YU, and work through the triangles YUS, SUT, TUW, WUV, and VUY, back to the same starting point. The final value must agree with the original, checking the calculations.

The azimuth say of a line VU near the centre can then be found astronomically, and if the survey is only a small one (so that we can assume a plane surface for the earth) ordinary rectangular co-ordinates will suffice, instead of choosing any particular map projection. Hence we choose any circuit starting with VU, and coming back to it, for example $VUTSYXZV$. We take the known azimuth of VU as the first bearing, and, knowing all sides and angles, we can calculate bearings, co-ordinates, etc., as in a closed traverse, except that here there can be no corrections. Any closing errors indicate mistakes in the calculation, which must be located and put right.

The bearings of the lines and co-ordinates of the stations are then used for checking details as described above.

For bigger triangulations, the principle is much the same, but modifications become necessary to allow for the shape of the earth. Full consideration of these is out of the question here, but we shall refer shortly to some of them.

Convergence of Meridians.

One modification, for instance, becomes necessary in consequence of the convergence of the meridians. We have assumed above that the direction of the north and south line remains parallel to itself, and that, in consequence, the *back* bearing of any line differs by 180° from its *forward* bearing.

Now, in reality, if AP (Fig. 32) be the meridian at A, and AB a straight line from A, then the meridian (or north and

south line) at B is in the direction BP, and is not, in general, parallel to AP, and the back bearing therefore does not differ from the forward by 180°.

Thus if we were to measure the true bearing or azimuth of AB at A (namely, the angle PAB), and from this and the included angle ABC (measured at B) we were to calculate the bearing of a line BC by our rules, this would not agree with the true bearing or azimuth of BC as measured at B (namely, the angle PBC).

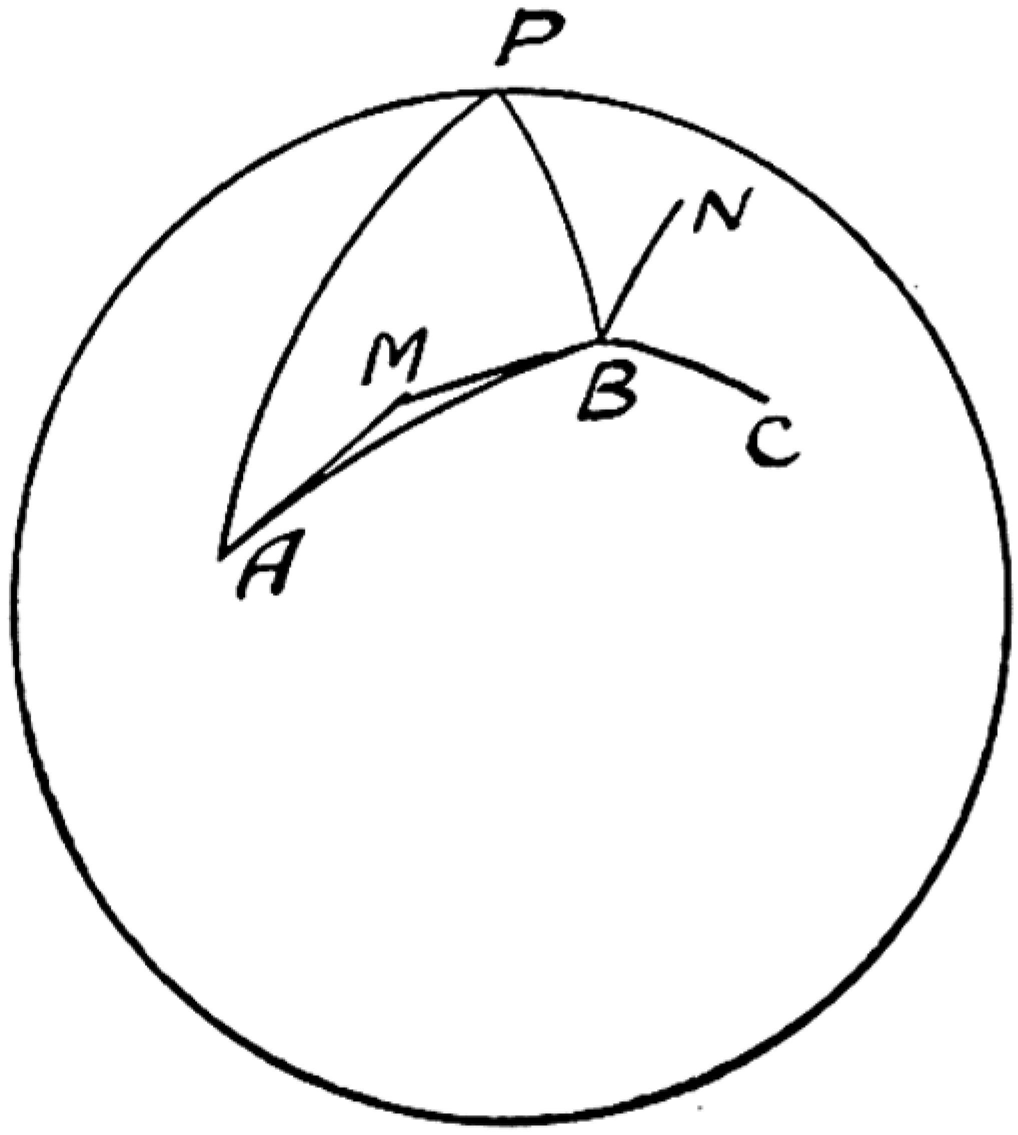


FIG. 32

If we take BN parallel to AP, the angle calculated would be NBC, and the angle NBP shows the discrepancy. It is the angle through which the meridian has turned in moving from A to B.

The matter is further complicated, however, by the fact that AB is an arc of a great circle, and not a true straight line. The direction of the line AB, as read from A, is that of the tangent AM, while the direction of BA, as read from B, is given by the tangent BM.

Hence in calculating convergence, we must take the change in direction of the line into account, as well as the angle PBN, and the definition of convergence is arrived at as follows: Measure the azimuth of AB at A, namely, PAB. To this add 180° (or subtract it, as the case may require). The result would give the bearing of BA at B, if there were no convergence. Now find the true azimuth of BA at B, and the difference is called the convergence of the meridians between A and B.

It is best calculated by spherical trigonometry, and, for those who are familiar with this branch of mathematics, a proof of the formula is given later (Chapter VI). The formula, however, is

$$x = \theta \sin \phi,$$

where x is the required convergence, and θ the difference of

longitude between A and B (both in the same units), and ϕ is the mean latitude of A and B.

It is assumed that the total range of latitude and longitude is small, say not more than 1° either way. An example of a calculation of latitudes, longitudes, and azimuths by the aid of this formula will be found in Chapter V, No. 4.

Levels in Triangulation.

If the mean surface of the earth were a flat plane, it would be a simple matter to find the difference of level between two

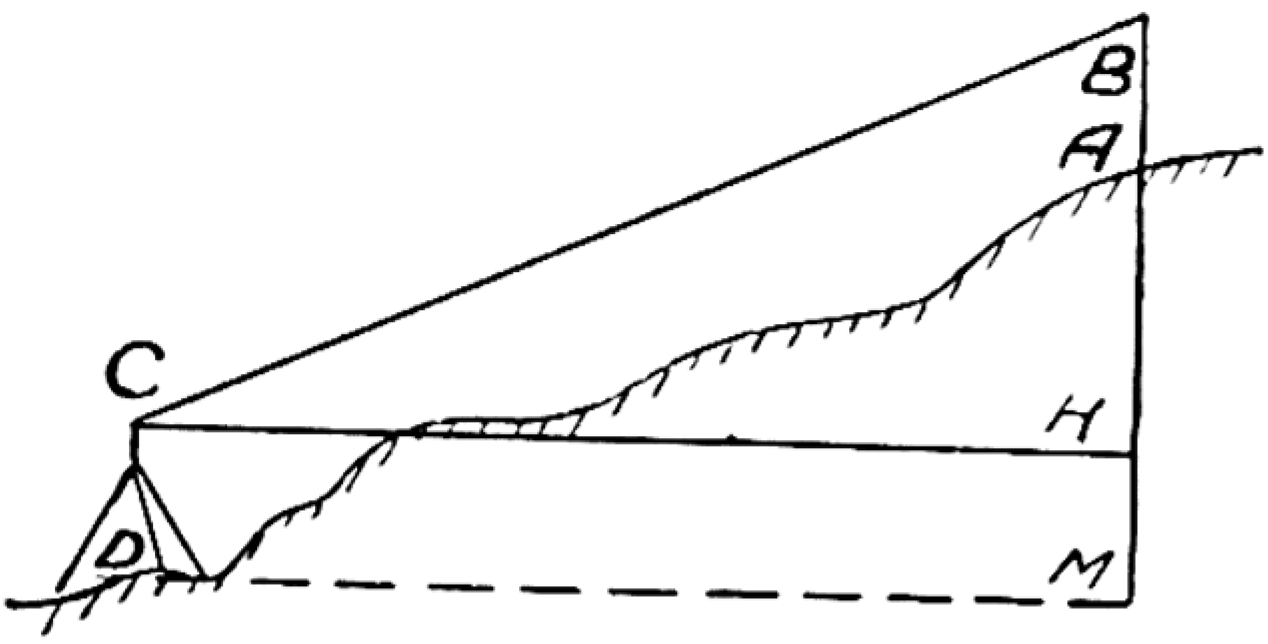


FIG. 33

triangulation stations, D and A (Fig. 33), by observing the vertical angles as well as the horizontal.

Let HCB be the angle of elevation observed from the theodolite at D to the signal at A.

Here DC = height of theodolite, AB = height of signal. The horizontal distance CH is found from the triangulation, as already described.

$$\begin{aligned} \text{Then rise } D \text{ to } A &= MA \\ &= HB + MH - AB \\ &= CH \times \tan HCB + MH - AB. \end{aligned}$$

Or, put into words,

$$\begin{aligned} \text{Rise} &= \text{distance} \times \tan \text{angle of elevation} \\ &\quad + \text{ht. of theodolite} - \text{ht. of signal.} \end{aligned}$$

Now, the earth's surface is *not* flat of course. Hence this simple calculation will not suffice for long distances, say over 1 mile.

Suppose the dotted line (Fig. 34) represents mean sea-level, and CN a surface parallel to it. This is called a *level* surface on the earth, and the rise from C to B is really represented by NB, *not* HB. Here CH is the horizon at C, from which the angle is measured. Now if we say that $BH = CH \tan BCH$, we are assuming that the vertical at B is perpendicular to the horizon at C. But the figure shows that this is not so. For distances not exceeding a mile, however, the angle CHB will lie between 90° and 90° 1′, and if HB be calculated exactly it differs little from $CH \tan BCH$. Even for a distance of 5 miles and angles of 5°, 10°, and 15° the error is about 1 part in 10,000, 5,000 and 3,000 respectively; in round numbers. Even the last figure is only equivalent to an error of about 15 sec. in the angle, or about 2 ft. in the height of a mountain about 7,000 ft. high, so that, for most purposes, we can take it that $BH = CH \tan HCB$, for distances not more than a few miles.

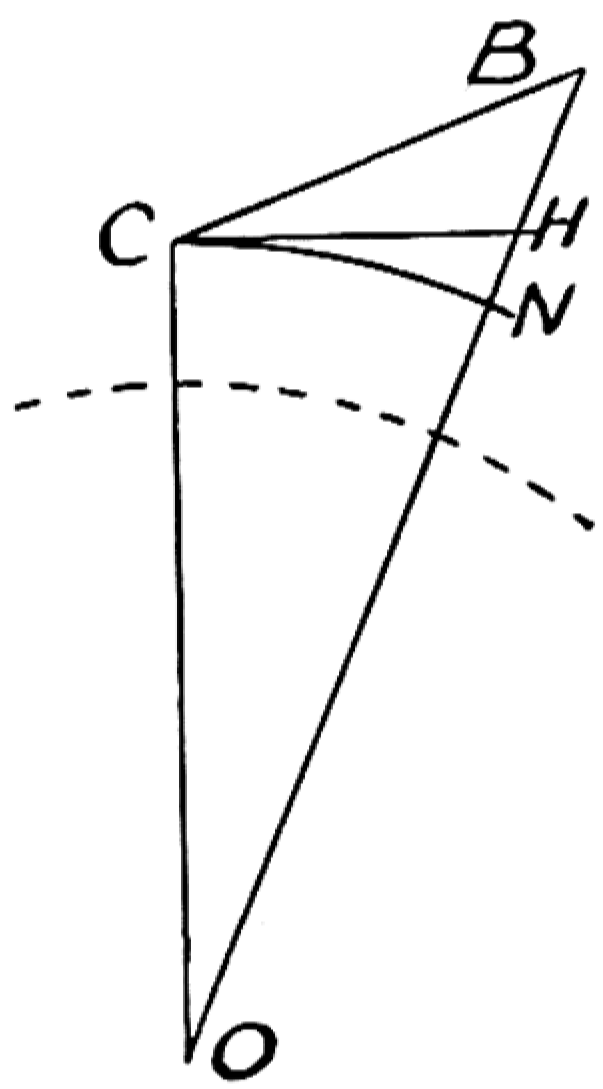


FIG. 34

We have still to find HN, however. This is called the curvature correction. If we assume that CN is a circular arc, we can find HN by a well-known proposition in Euclid, whereby

$HC^2 = HN\,(HN + 2\,NO) = HN \times 2\,NO$ very nearly, as HN is small.

$$\text{Hence } HN = \frac{HC^2}{2\,NO} = \frac{(\text{distance})^2}{\text{earth's diameter}}.$$

For a distance of 1 mile, this works out to about 8 in.; for other distances it is proportional to distance squared. Thus at $\frac{1}{2}$ mile it would be 8 in. $\times (\frac{1}{2})^2 = 2$ in. At 5 miles it would be 16 ft., and so on.

For exact work the matter is further complicated by the fact that the rays of light do not travel in straight lines, but undergo refraction as already explained (p. 14). The amount of this refraction effect is very variable, so that atmospheric conditions must be favourable for a reliable result. But as a rough mean, it is usual to take it at about *one seventh* of the curvature effect, but in the *opposite direction*.

True rise C to $B = NB = BH +$ curvature.

Thus curvature is *plus* in calculating a rise.

Refraction makes the angle of elevation (as a rule) appear too great. Hence it gives too high a value for BH, and the correction is *minus*. Altogether, then,

$$\text{Rise} = \text{dist.} \times \tan \text{angle of elevation} + \text{curvature} \\ - \text{refraction} + \text{ht. of theodolite} - \text{ht. of signal.}$$

Where the angle is an angle of *depression*, the first term will be *minus*. The final result if *plus* indicates a *rise*, if *minus* a *fall*.

For a worked example, see Chapter V.

We may deal with this calculation in another way.

In Fig. 35, suppose the circle represents the level surface (parallel to mean sea-level), through C, and that we have observed the angle of elevation, HCB to B.

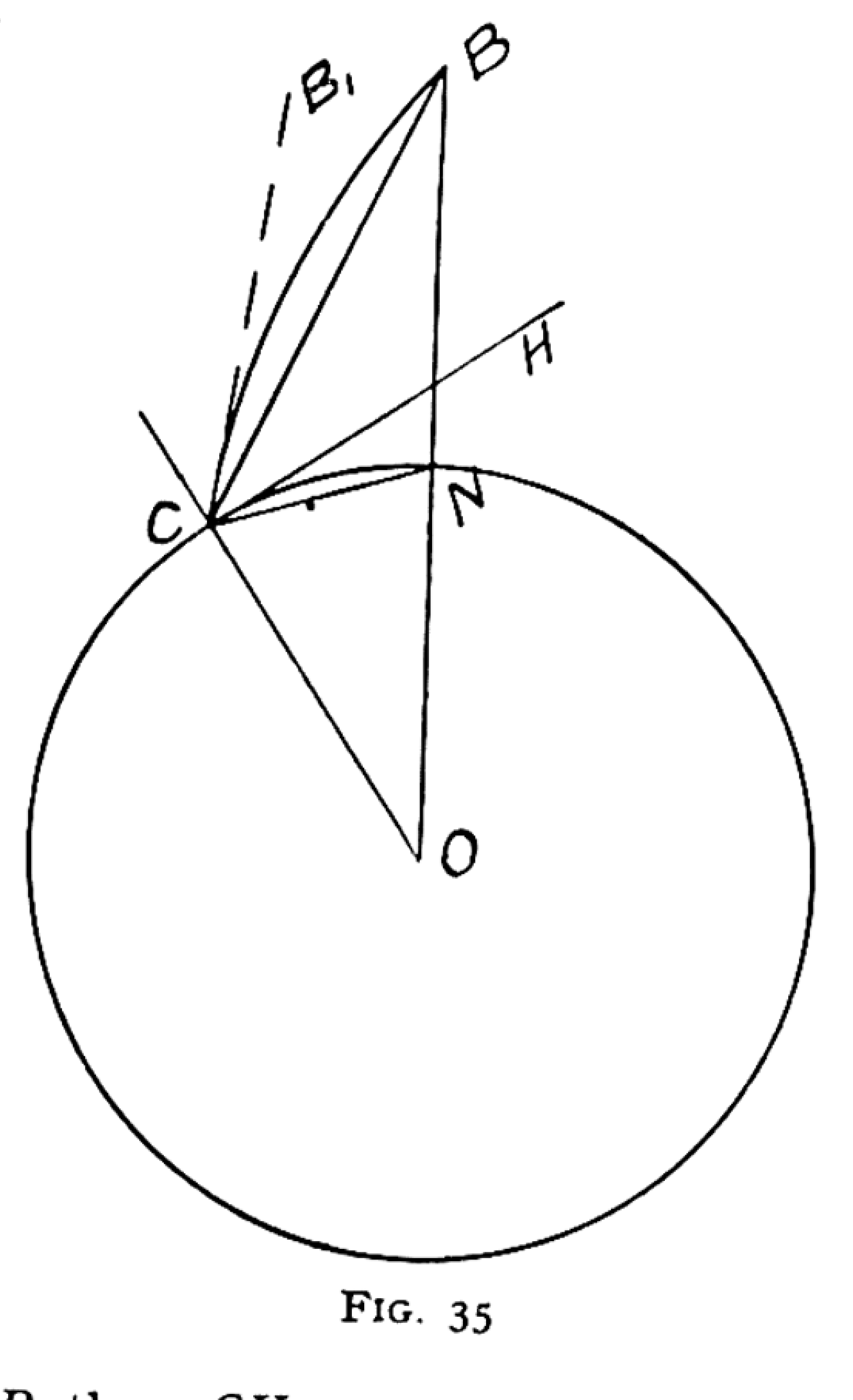


FIG. 35

In these figures, both the height of B and the distance CH are, of course, enormously exaggerated compared with the earth's radius.

Now join CN.

As we have seen, NB is the true rise from C to B, and as CN is even more nearly perpendicular to OB than CH, we may safely say $NB = CN \tan NCB$.

Now in any practical case CN is practically equal to the horizontal distance between A and B, hence if we can find the angle NCB, we can find NB at once.

Now $NCB = HCB + NCH$; and NCH is the angle between a tangent CH, and a chord CN, and is therefore equal

to *half* the angle CON subtended at the earth's centre by the chord, that is by the line CN.

Again, if the earth be regarded as a sphere of radius R, its circumference $= 2\pi R$, and the length of arc which subtends an angle of *one second* at the centre is $\dfrac{2\pi R}{360 \times 60 \times 60}$.

With the general mean radius on page 49, this works out to about 101·3, which figure may be slightly modified by the actual shape of the earth.

Hence if we divide the horizontal distance CN (which is known from the triangulation) by this figure, we shall obtain the angle CON, and *half* this gives the angle HCN to be added to the angle of elevation, HCB.

In consequence of the curved path of the light rays from B to C, the angle of elevation is actually observed as B_1CH, where B_1CB is the error due to refraction.

The mean effect of this, as has been stated, is roughly one-seventh of the curvature, and this may also be applied as a correction to the angle, and is clearly a *minus* correction.

Hence, altogether—

(1) Observe the angle of elevation or depression. Call it *plus* if elevation, *minus* if depression.

(2) Divide the horizontal distance in feet by the arc of 1 sec., which is about 101·3 ft., and take *half* the result as the curvature correction in seconds. Call this *plus*.

(3) Take one-seventh of the last result for refraction, and call it *minus*.

Add (1), (2) and (3) algebraically. The result if plus will indicate a *rise*, and if minus a *fall*, and if A be the sum, the rise or fall will be distance $\times$ tan A.

Resection in Triangulation.

Many interesting problems arise in the course of surveys involving triangulations. Amongst these is the problem of fixing a station by observations *from* it to three known triangulation stations.

This has been referred to in Vol. I, in connection with the plane-table and in hydrographic surveys, under the head of *resection*. We shall therefore proceed to show how the same problem is solved in triangulation.

Let A, B, C (Fig. 36) be three triangulation stations in the field, say 2 to 5 miles apart. There may be little or no detail to show in their immediate vicinity, but about a mile or two away there may be some important detail which could well be surveyed, say by a traverse starting at a station X. The latter must, of course, be connected with the triangulation. We *could* do this by starting, say, from B, and surveying over the mile or more of intervening space. But it is clearly quicker (and in most cases more accurate) to avoid this, and (having chosen X so that A, B, C are all visible from it) to fix its position by resection.

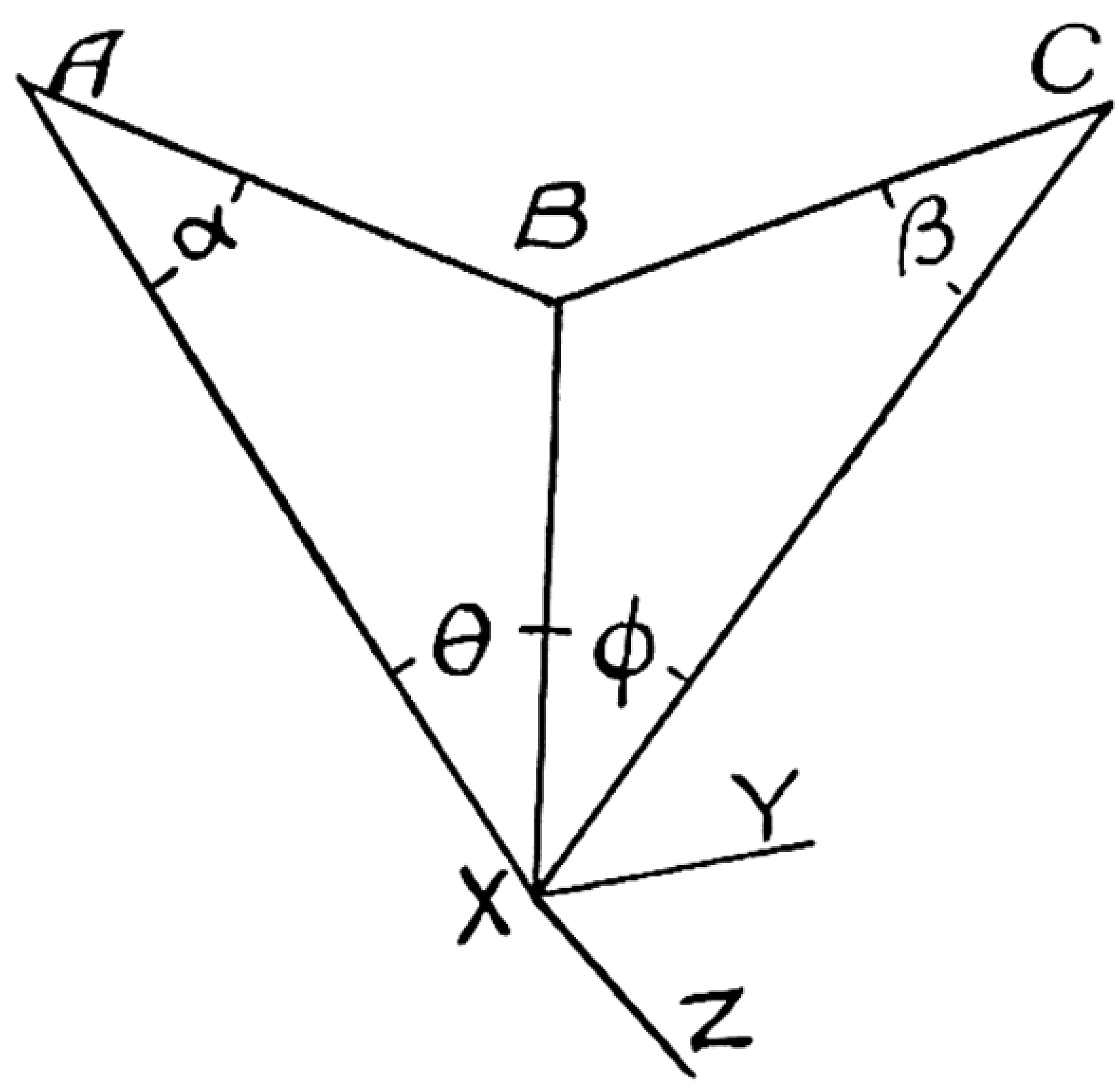


FIG. 36

The problem is, then, that the lengths and bearings of AB, BC, the angle ABC, and the co-ordinates of A, B and C, are all known from the triangulation. We require to find the co-ordinates of X.

Set up the theodolite and X, and read the angles AXB, BXC, which we have marked θ and ϕ respectively. If XY and XZ are lines of the required traverse, read at the same time the angles YXZ and CXY.

Now the angles BAX and XCB, which we have marked α and β, are unknown, but in the triangle ABX we have

$$\frac{AB}{BX} = \frac{\sin \theta}{\sin \alpha} \quad . \quad . \quad . \quad . \quad (1)$$

and in the triangle CBX,

$$\frac{BX}{CB} = \frac{\sin \beta}{\sin \phi} \quad . \quad . \quad . \quad . \quad (2)$$

Now by multiplying these equations together, we obtain

$$\frac{AB}{CB} = \frac{\sin \beta}{\sin \alpha} \cdot \frac{\sin \theta}{\sin \phi}$$

Here AB, CB, θ, and ϕ are all known; hence we rearrange the equation to put these known quantities on one side, keeping the unknowns, α and β, on the other side.

$$\therefore \qquad \frac{\sin \beta}{\sin \alpha} = \frac{AB}{CB} \cdot \frac{\sin \phi}{\sin \theta}$$

The value of the right-hand expression can be computed, as all the quantities are known; call it k.

$$\text{Hence } \frac{\sin \beta}{\sin \alpha} = k, \text{ where } k \text{ is known} \qquad . \quad . \quad . \quad (3)$$

Next we shall show that the sum of $\alpha + \beta$ is known though we do not know the separate values.

Now $\qquad \beta + \phi + CBX = 180^\circ$

and $\qquad \alpha + \theta + XBA = 180^\circ$.

$\therefore \alpha + \beta + \theta + \phi + CBX + XBA = 360^\circ$.

But $CBX + XBA = CBA$, the angle between CB and BA, as measured *on the side facing* X. This is known from the triangulation.

Hence $\alpha + \beta = 360^\circ - CBA - (\theta + \phi)$

$= A$, say, where A is known, as all the quantities on the right are known.

Hence we know the sum $\alpha + \beta$, and the ratio $\frac{\sin \alpha}{\sin \beta}$. Many methods of finding α and β from these have been evolved, but the following will suffice.

We have $\beta = A - \alpha$; $\therefore \sin \beta = \sin A \cos \alpha - \sin \alpha \cos A$.

Substitute in equation No. 3.

$$\therefore \qquad \frac{\text{Sin } A \cos \alpha - \sin \alpha \cos A}{\sin \alpha} = k,$$

or $\cot \alpha \sin A - \cos A = k$.

$$\therefore \text{Cot } \alpha = \frac{k + \cos A}{\text{Sin } A} = k \operatorname{cosec} A + \cot A.$$

Hence α can be found, and therefore β.

Now knowing α and θ, and AB, we can solve the triangle ABX to find AX. Knowing the bearing of BA, and the angle α, we can find the bearing of AX by our usual rules, and

similarly knowing the co-ordinates of A and the bearing and length of AX, we can find co-ordinates for X, as in traversing.

The whole of this calculation should be repeated in the triangle XBC, to find the co-ordinates of X from those of C, as a check.

Then knowing the bearing of CX and the angle CXY, we can find the bearing of the first traverse line XY, and proceed to make, say, a closed traverse round the detail.

For the dimensions assumed we can neglect the earth's curvature. Indeed it is seldom necessary to consider this in connection with this problem.

We have shown in Vol. I that the method fails if X lies on the circumference of the circle through ABC; if it lies *near* this circumference, the method becomes inexact. Hence care must be taken to choose stations for the observation which are clearly *not* concyclic with X.

CHAPTER IV

PHOTOGRAPHIC SURVEYING

Ground Photography.

PHOTOGRAPHY is now so much used in surveying that it is impossible to conclude this section without some reference to its application for this purpose.

Photographic surveying must be considered from two different points of view. That is, the photographs may be taken either with cameras fixed on ordinary stands on the ground, or they may be taken from aeroplanes, or other aircraft.

We shall consider, firstly, ground photographs, which are usually taken with an instrument called a photo-theodolite.

Photo-theodolite.

This (in one form) is shown in Fig. 37, and consists essentially of a camera on an ordinary theodolite mounting, so that, by reading the horizontal circle, we can tell the direction in which the axis of the camera is pointing. The camera also contains the following special features, all of which can be brought into contact with the plate before exposure, so that they come out on the photographs—

(1) Vertical and horizontal hairs, the intersection of which lies on the axis of the camera.

(2) Small changeable celluloid plates, on which we can write identification numbers, etc.

(3) A celluloid graduated circle attached to a compass, so that the magnetic bearing of the camera axis is automatically printed, and serves as a check on the horizontal angle.

The camera box is fitted with levels, so that the axis can be brought truly horizontal.

The focal length (that is to say, the perpendicular distance of the centre of the lens from the photographic plate during exposure) is known, the value being supplied by the makers.

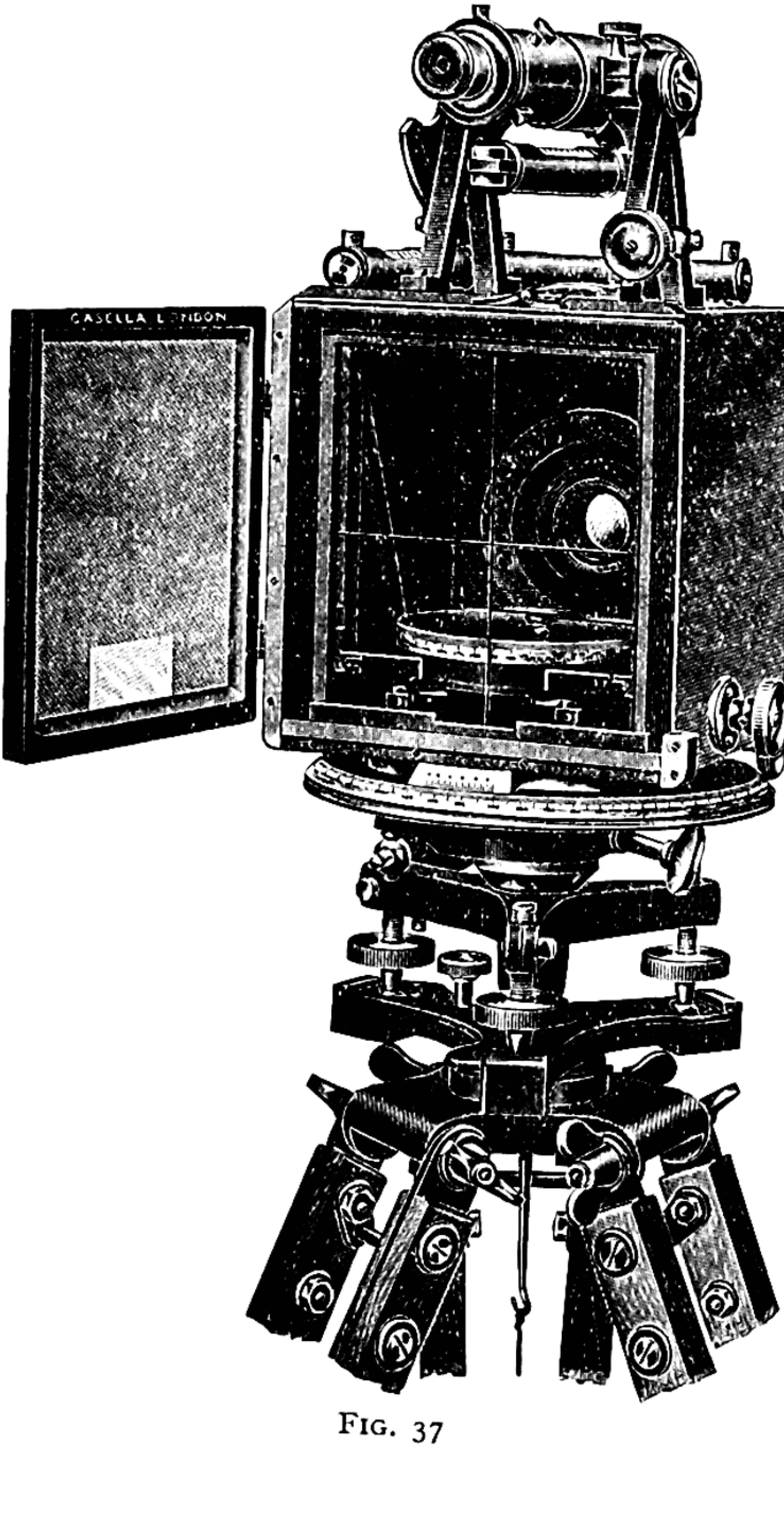


FIG. 37

Method of Work.

To carry out a survey by this method, a base line AB is chosen (so as to overlook the ground to be surveyed) and measured. The instrument is set up at A, directed along the base line, and the horizontal circle is read. The camera is then turned to such a direction that a photograph taken in that position will show as many as possible of the points to be mapped. Another reading of the horizontal circle will give the horizontal angle (say θ) from the base line.

The photo is then taken. Referring to Fig. 38, suppose

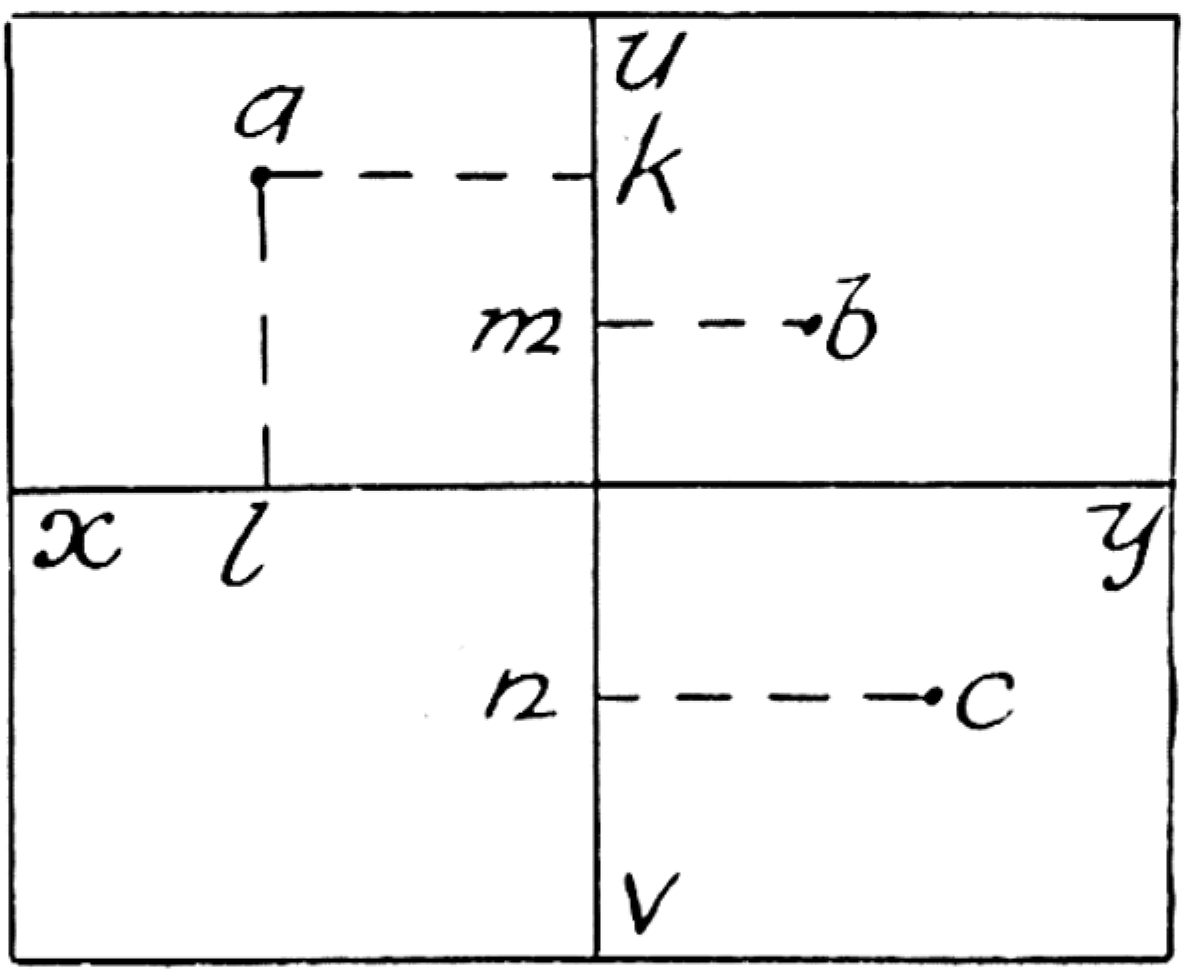


FIG. 38

xy, uv, are the horizontal and vertical hairs, and a, b, c are the photographs of three points to be plotted.

It will be clear that we can measure from the negative the distances ak, bm, and cn of these points from the vertical hair.

Now in Fig. 39, let AB be the base line, plotted to scale on the paper. Set off the measured angle θ between it and the camera axis, and draw the line OAO_1, representing the axis. Suppose that along this line we set off OA to represent the known focal length (*full size*) and draw Oa perpendicular to OA. Then Oa will represent the plate, and we can set off Oa $= ak$ (Fig. 38), $Ob = bm$, and so on (also full size). The point A represents the centre of the lens (through which rays of light travel in straight lines), so we join aA, bA, cA, and produce as shown. Then the plans of a, b, c must lie somewhere on these lines, respectively.

It is clear that if another photograph be taken from B (showing the same points) and treated in the same way, we shall obtain *two* lines on which each point must lie, and their intersection will fix the point.

A large number of points may be fixed in this way from a few photographs.

Actually, instead of setting off the focal length AO *behind*

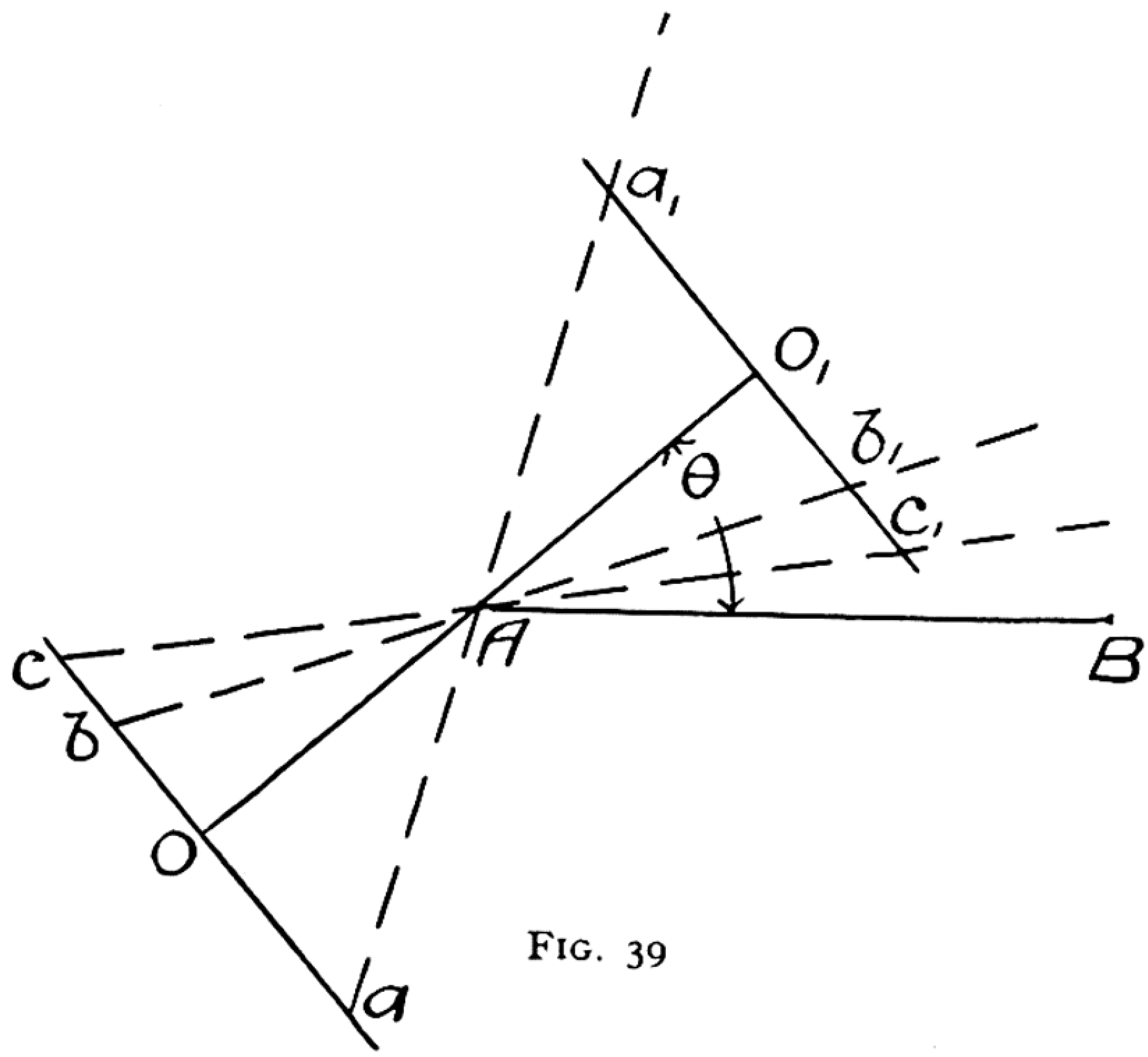


FIG. 39

the station as shown at AO (which is its actual position in the field) we set it off *forwards* as shown at AO_1, and reverse the directions of Oa, Ob, etc., as shown at O_1a_1, O_1b_1. This means that these distances are set off on the same side as they appear in the photograph.

Theoretically, the two methods come to the same thing; but practically it means that lines have not to be produced so far, and we have shown in Vol. I that it is inaccurate to produce short lines for long distances.

Levels.

Besides the plans, however, we can find the levels of the points.

The horizontal hair (xy, Fig. 38) is at the same level as the centre of the lens. It follows that all points whose photographs are on xy will lie on a contour line at the same level as the camera; all points (like a, b, Fig. 38) which lie above the hair will be higher than the camera, and those below it will be lower.

Now suppose we measure the height al (Fig. 38) of the point a above the hair, and let a (Fig. 40) be the final plan of that point, as fixed from A and B.

Aa_1 gives the *horizontal* distance inside the camera, from the lens to the image of a; al tells how much the ray of light

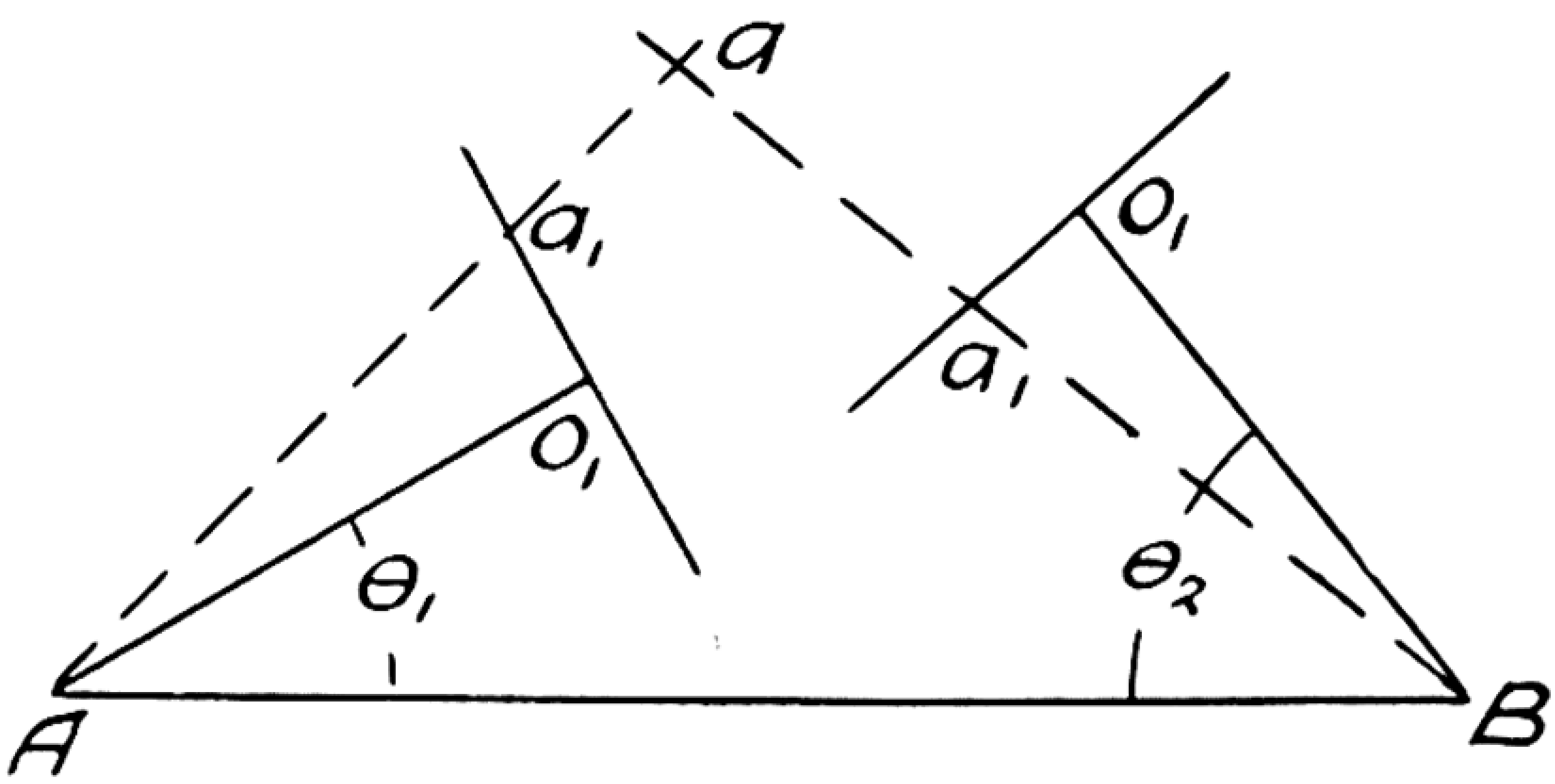


FIG. 40

rises in that distance; Aa, measured to scale in feet, tells the actual distance in feet from the lens to the point on the field. Hence, if h be the height of the point above camera level, it is clear that the ray of light rises a height h in the distance Aa; hence by proportion $\frac{h}{Aa} = \frac{la}{Aa_1}$. Here Aa is measured in feet to scale; la and Aa_1 are full size in any convenient unit (e.g. both in centimetres) and the result, h, is in feet. If we add the height of the camera, we shall obtain the height of the point above the station A.

Limitations.

This method is exceedingly quick, so far as fieldwork goes, on suitable ground. But generally vegetation, buildings, undulating ground, and other obstructions intervene, making it almost impossible to find stations from which all the points required can be seen or photographed. This seriously limits

the application of the method, and has led largely to the development of surveying by *aerial* photography.

Aerial Photographs.

By taking the photographs from the air, it is obvious that these obstructions become less important, and far more ground can be covered clearly in one exposure.

Air photographs are of two kinds, (1) oblique, (2) vertical. Oblique photographs are taken with the axis of the camera more or less horizontal, and plotting from them is carried out somewhat on the same principles as from ground photographs. But there are some important differences. For instance the exact "station" from which the photograph was taken is unknown, and not only is the camera axis not horizontal, but its exact direction is unknown, either horizontally or vertically.

These difficulties can be overcome by surveying a limited number of points by ground methods, so that each photograph shall show four or more points whose plans are known. These, by the aid of certain optical appliances, enable us to put the plate in the same position with regard to the plan that it occupied in the field.

Vertical photographs are intended to be taken with the axis of the camera vertical.

Now it will be clear that if a piece of ground is a perfectly level plane surface, and we photograph it from the air on to a plate which is also truly horizontal, and therefore parallel to the ground, the negative will give us an exact plan or map of the ground to a definite scale depending on the ratio of the focal length AO (Fig. 41) to the height (AN) of the aeroplane above the ground. Thus, in Fig. 41,

$$\frac{be}{BE} = \frac{AO}{AN}$$

Suppose we have a camera with a focal length of 8 in. How high must the machine fly, in order to give a scale of 6 in. to 1 mile?

Here $AO = 8$ in.; and if $BE = 1$ mile, be is to be 6 in. We require AN.

Clearly $$\frac{AN}{BE} = \frac{AO}{eb}$$

or $AN = \frac{8}{6} \times 1$ mile $= 1\frac{1}{3}$ miles. Actually they might be

taken from a greater height (and subsequently enlarged) for cheapness.

If the earth were a perfectly flat level plain, it is clear that a sufficient number of such photographs joined together would give at once a complete map of the desired scale; and all points would be visible unless they were actually *under* some other object, such as a tree.

Fortunately, or unfortunately, the earth is *not* a plane level surface, and this somewhat complicates the matter.

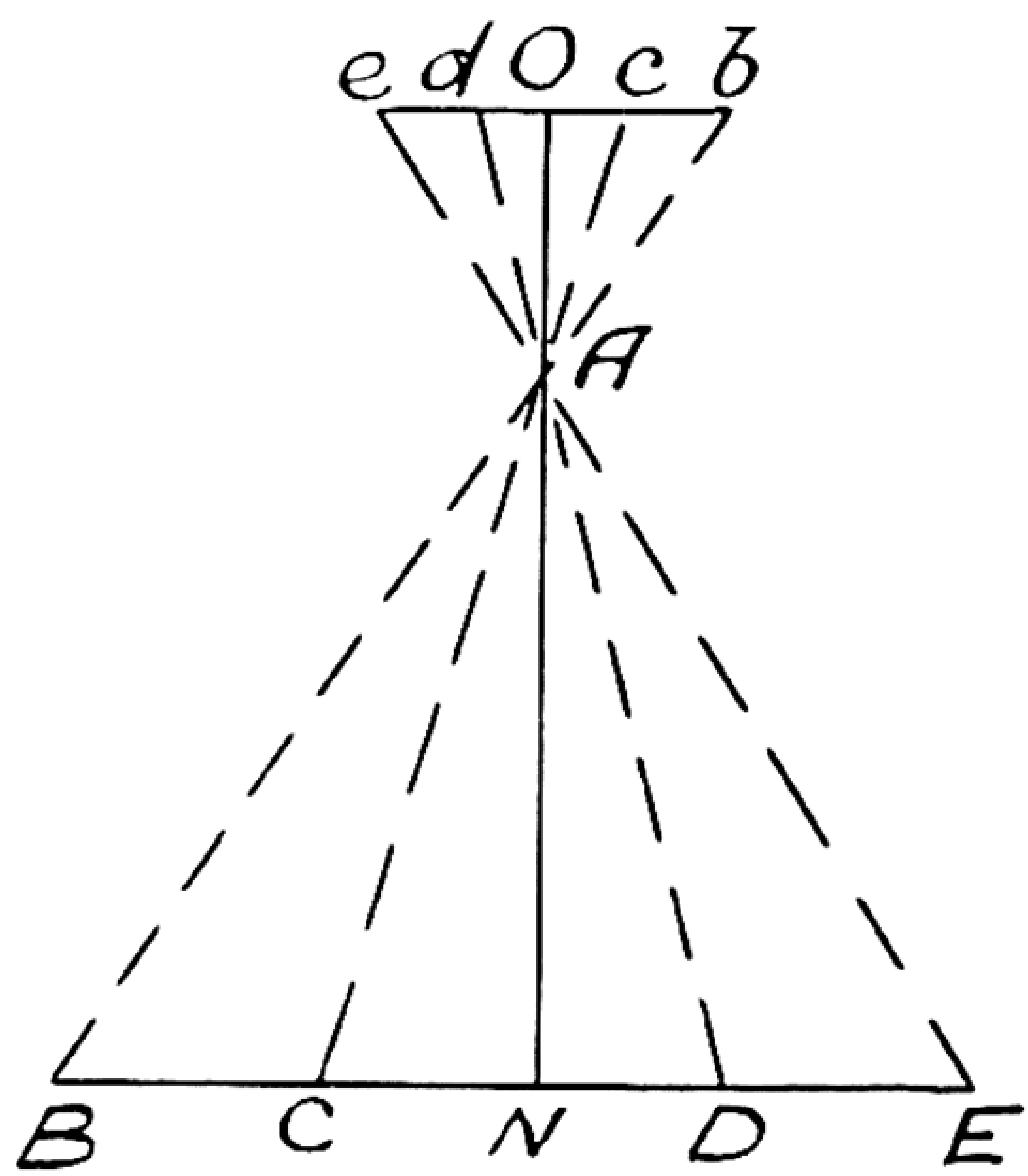


FIG. 41

Moreover, no means have yet been devised whereby the axis of the camera may be kept exactly vertical on a rapidly moving aeroplane; nor is it possible to say with certainty that the machine is exactly at the right height.

To overcome completely variations in the ground level, each point must be shown on at least two photographs; and to overcome variations in the direction of camera and height of plane, and to fix the exact position from which the photograph was taken, we must resort, as before, to methods of ground surveying to fix a limited number of points, some of which are shown on each photograph.

As before, these enable us to replace the plate in its proper position with respect to the plan, and the photographs can be rectified accordingly.

A general level is chosen for the plane of the plan. By having each pointer on two photographs, if the rays to a point, as found from the two images, meet *below* the plan level, we know that the point is below the chosen level, and vice versa.

A full account of the appliances for carrying out this rectification is beyond the scope of this book. For that, together

with a general account of the whole subject and of its applications to geography teaching and other purposes, the student is

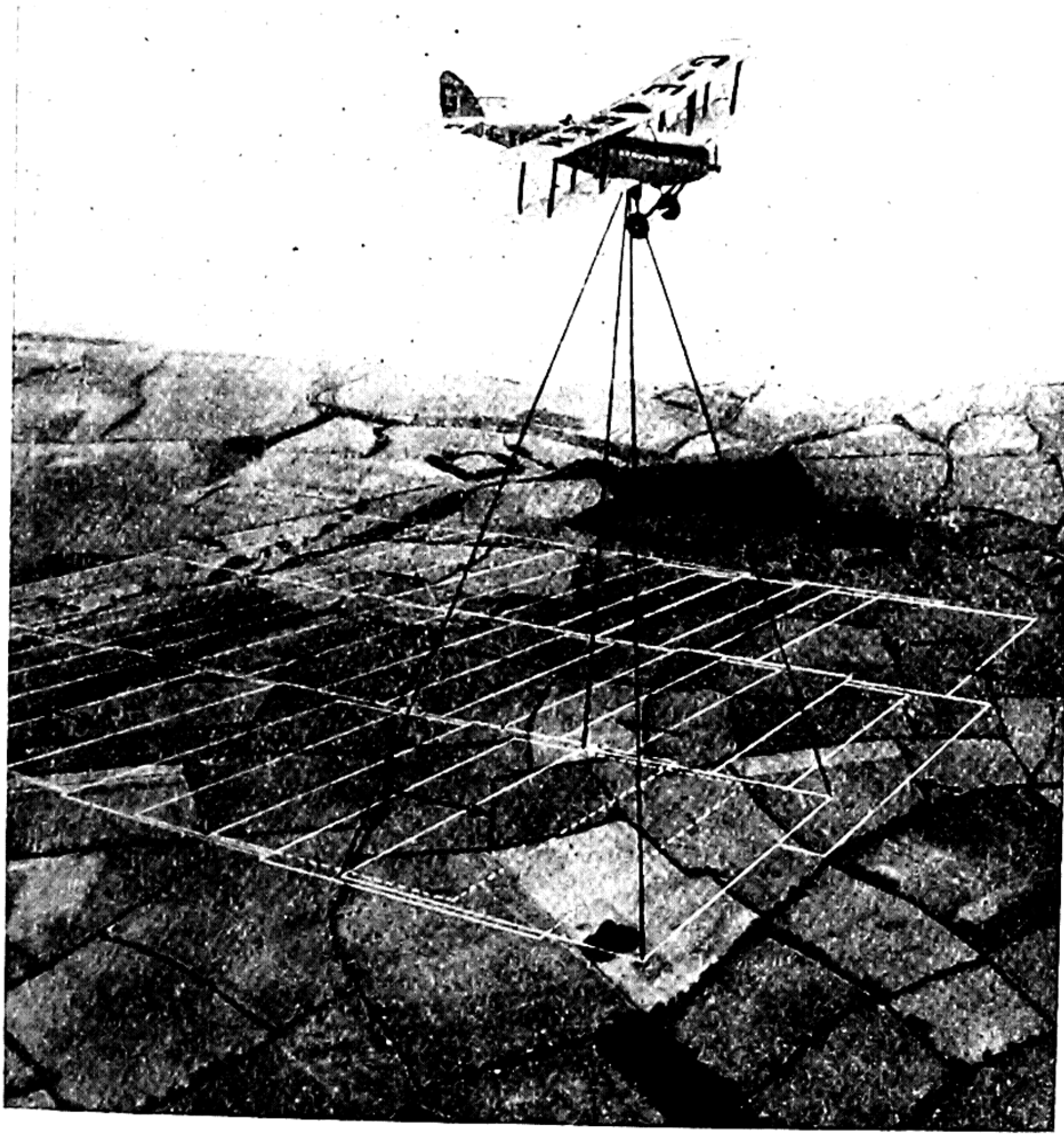


FIG. 42

referred to *Aerial Photography*, by Winchester and Wills (Chapman and Hall).

The writers are indebted to the publishers of this book for the use of the diagram in Fig. 42, to show the method of taking aerial photographs for surveying. It will be seen that considerable overlap is allowed, not only to obtain points on two photographs, but also to ensure that the whole area is covered, as it is impossible to fly in a perfectly prearranged straight line.

CHAPTER V

WORKED EXAMPLES

WE shall conclude this section with a few worked examples.

(1) The co-ordinates of two triangulation stations, A, B, are given in the table, and also the lengths and bearings of three traverse lines, AK, KL, LB, connecting them.

Calculate the corrected co-ordinates of K and L.

Pt.	Co-ordinates	
	N.	E.
A	6108·2	2168·0
B	3734·6	1982·7
Line	**Bearing**	**Length**
	° ′	
AK	210 47	852·7
KL	130 54	706·2
LB	193 36	1216·0

Reduced bearings :

$$AK = 30^\circ\ 47'\ \text{S.W.}$$
$$KL = 49^\circ\ 6'\ \text{S.E.}$$
$$LB = 13^\circ\ 36'\ \text{S.W}$$

Difference of latitude and departure for AK—

$$\begin{array}{rlrl} \log\cos 30^\circ\ 47' = & 9{\cdot}934048 & \log\sin = & 9{\cdot}709094 \\ \log 852{\cdot}7 = & \underline{2{\cdot}930796} & & \underline{2{\cdot}930796} \\ & 2{\cdot}864844 & & 2{\cdot}639890 \end{array}$$

$$\begin{array}{rlcl} \text{Diff. of lat.} = & 732{\cdot}6\ \text{S., departure} & = & 436{\cdot}4,\ \text{W.} \\ \text{Similarly for } KL & 462{\cdot}4,\ \text{S.,} & \text{and} & 533{\cdot}8,\ \text{E.} \\ \text{and for } LB & \underline{1182{\cdot}1},\ \text{S.,} & ,, & \underline{285{\cdot}9},\ \text{W.} \\ \text{Totals from } A \text{ to } B, & \underline{\underline{2377{\cdot}1}},\ \text{S.,} & ,, & \underline{\underline{188{\cdot}5}},\ \text{W.} \end{array}$$

Note that to find the total departure we take the *difference* between the easts and wests. The result is west because the wests are more than the easts.

Now taking the difference between the original co-ordinates of A and B, we have

$$\begin{array}{rll} & \text{N.} & \text{E.} \\ A = & 6108{\cdot}2 & 2168{\cdot}0 \\ B = & \underline{3734{\cdot}6} & \underline{1982{\cdot}7} \\ & 2373{\cdot}6 & 185{\cdot}3 \end{array}$$

The results from the traverse must agree with these. We see that the southings from the traverse are 2377·1, instead of 2373·6, or *too much* by 3·5 ft.; and similarly the departures by the traverse are too great by 3·2 ft.

The sum of the lengths in the table is 2774·9. Hence, by Bowditch's rule, the correction to difference of latitude for AK is $3{\cdot}5 \times \frac{853}{2775}$, say 1·1.

Similarly for KL it is $3{\cdot}5 \times \frac{706}{2775}$, say 0·9

and for LB $\quad 3{\cdot}5 \times \frac{1216}{2775}$, say 1·5.

The student must not waste time at examinations working these out with meticulous care. One says that 853 is rather less than one-third of 2,775, so takes 1·1 for the first, and so on. The last one must have such a value as to make the total right, and we must see that it comes *about* right by the above rough rule.

All these corrections are to be *subtracted*, as the southings from the traverse were too big.

Similarly the corrections in departure are—

For AK, $3{\cdot}2 \times \frac{853}{2775}$, say 1·0.

,, KL, $3{\cdot}2 \times \frac{706}{2775}$, say 0·8.

,, LB, $3{\cdot}2 \times \frac{1216}{2775}$, say 1·4.

Total 3·2.

Here we must *decrease* the westing and *increase* the easting, so as to *decrease* the total westing. Thus the corrected differences of latitude and departure are—

Line	Latitude	Departure
AK	732·6 − 1·1	436·4 − 1·0
KL	462·4 − 0·9	533·8 + 0·8
LB	1182·1 − 1·5	285·9 − 1·4

		N.	E.	
Co-ordinates :	A,	6108·2	2168·0	
		731·5, S.	435·4, W.	
	K,	5376·7	1732·6	
		461·5, S.	534·6 E.	
	L,	4915·2	2267·2	
		1180·6, S.	284·5, W.	
	B,	3734·6	1982·7,	which checks.

(2) The top, C, of a mountain is observed with a theodolite from two triangulation stations, A and B, the horizontal distance AB being 26,400 ft., A being 7,580 ft. above sea-level. The observed horizontal angles are $CAB = 84^\circ\ 16'$, $ABC = 81^\circ\ 26'$. From A, the elevation to the top of the mountain is found to be $5^\circ\ 54'$ face right, $5^\circ\ 56'$ face left. Calculate (*a*) the horizontal distances AC, BC, and (*b*) the height of C above sea-level. Assume that one second of arc on the earth = 102 ft.

$$\begin{array}{rl}
\text{Here } CAB = & 84^\circ\ 16' \\
ABC = & 81^\circ\ 26' \\
\hline
& 165^\circ\ 40' \\
\therefore BCA \quad & 14^\circ\ 20', \text{ neglecting spherical excess.} \\
\hline
& 180^\circ\ \ 0'
\end{array}$$

$$\therefore AC = 26400 \times \frac{\text{Sin } 81^\circ\ 26'}{\text{Sin } 14^\circ\ 20'},\ BC = 26400 \times \frac{\text{Sin } 84^\circ\ 16'}{\text{Sin } 14^\circ\ 20'}$$

$$\begin{array}{lrl}
\text{Log } 26400 & = & 4{\cdot}421604 \\
\text{,, } \sin 14^\circ\ 20' & = & 9{\cdot}383685 \\
\hline
& & 5{\cdot}037919 \\
\text{,, } \sin 81^\circ\ 26' & = & 9{\cdot}995127 \\
\hline
& & 5{\cdot}033046 \\
& & AC = 107{,}906 \text{ ft.}
\end{array}
\qquad
\begin{array}{lrl}
& & \bar{5}{\cdot}037919 \\
\text{Log } \sin 84^\circ\ 16' & = & 9{\cdot}997822 \\
\hline
& & 5{\cdot}035741 \\
& & BC = 108{,}578 \text{ ft.}
\end{array}$$

The mean angle of elevation, AC, is $5° 55'$.
(*a*) Without correcting the angle,
Calculated rise $= 26400 \times \tan 5° 55' = 2{,}736$ ft.
The distance is exactly 5 miles.
Hence curvature correction (approx.) $= 8 \times 25$ in.

$$\begin{array}{lrl} & = 16{\cdot}7 \text{ ft.}, & \textit{plus} \\ \text{Refraction correction, } \tfrac{1}{7}\text{th of this} & = 2{\cdot}4 \text{ ,,} & \textit{minus.} \\ \hline & 14{\cdot}0 \text{ ft., say,} & \textit{plus.} \end{array}$$

Hence the corrected rise is 2,750 ft.

It must be assumed that the level given for A includes the height of the theodolite, and that the ground at C is observed.
Hence level of $C = 7580 + 2750 = \underline{\underline{10{,}330 \text{ ft.}}}$

(*b*) If we correct the angle, we have to find the angle at the earth's centre for the distance 26,400 ft.

If $x =$ No. of seconds, we have

$$x = \frac{26{,}400}{102}, \text{ as 102 ft. subtend one second.}$$

$$\begin{array}{lrl} x = 259'' \text{ nearly, } \tfrac{1}{2} & = 129'' & = \text{curvature correction,} \\ \text{less one-seventh} & = 18'' & = \text{refraction} \\ \hline & 111'' & = \text{correction to be added.} \end{array}$$

$$\begin{array}{lr} \text{Observed angle} = & 5° \ 55' \\ + & 1' \ 51'' \\ \hline \text{Corrected angle} = & 5° \ 56' \ 51''. \end{array}$$

Rise $= 26{,}400 \times \tan 5° 56' 51''$.
$= 2{,}750$ ft., as before, to the nearest foot.

(3) If the latitudes of two stations are found to be $49° 57'$ N. and $51° 13'$ N. respectively, while the meridian distance between them is 462,000 ft., find the average radius of curvature of the meridian between those latitudes.

Also if the length of one degree of parallel in lat. $50°$ N. is found to be 235,220 ft., find the radius of the parallel (to the nearest 1,000 ft.) and the radius of the earth regarded as spherical (London University, Cartography, 1927).
Difference of latitude $= 51° 13' - 49° 57' = 1° 16'$.

This is the angle subtended, at the centre of curvature, by an arc or distance of 462,000 ft.

Hence if R be the radius of curvature in feet, we have

$$\frac{1^\circ\ 16'}{180^\circ} = \frac{462{,}000}{\pi R}$$

$$\therefore\ \pi R \times \frac{76}{60} = 180 \times 462{,}000, \text{ whence } R = \frac{60 \times 180 \times 462{,}000}{\pi \times 76}$$

The solution by logarithms is appended. With fractions having more than one term in the denominator, always begin with the denominator. Add the logarithms of its factors, and *subtract* the result from zero. This is called the *colog*. Then add the logarithms of the factors of the numerator, and tabulate the whole as shown.

No.	Log.
π	·4971499
76	1·8808136
	2·3779635
Colog	$\bar{3}$·6220365
60	1·7781513
180	2·2552725
462,000	5·6646420
	7·3201023

Hence $R = 20{,}898{,}000$ ft.

If one degree of parallel in 50° N. = 235,220 ft., and radius of parallel $= r$,

$$2\pi r = 360 \times 235{,}220$$

$$\therefore\quad r = \frac{360 \times 235{,}220}{2\pi} = 13{,}477{,}000 \text{ ft.}$$

And if $\quad \rho =$ radius of earth, regarded as spherical,

$$r = \rho \cos 50^\circ,$$

hence $\quad \rho = r \sec. 50^\circ = 20{,}918{,}000$ ft.

(4) Referring back to Fig. 32, page 66, suppose the latitude and longitude of a triangulation station A have been observed as 54° 21′ 15″ N. and 2° 21′ 14″ W. respectively, and the azimuth and length of AB as 68° 27′ 20″ east of north, and 48,762 ft. respectively. Also the clockwise angle ABC has been observed as 204° 27′ 18″.

Find the latitude and longitude of B, and the azimuth of BC, assuming the formula for convergence on page 66.

The first step is to find the "difference of latitude" and departure by the ordinary traverse rules—

Diff. of lat. = 48,762 cos 68° 27′ 20″ = 17,907 ft.
Departure = 48,762 sin 68° 27′ 20″ = 45,355 ft.

To continue the calculation conveniently, more extended tables of the lengths of arcs of meridian and parallel are required than those given in Chapter IX. Such tables will be found in *Topographical Surveying*, by Col. Close, and elsewhere, and from them we extract the following information—

Latitude	One Second of Meridian	One Second of Parallel
54° 20′	101·4461	59·2874
25′	101·4476	59·1675

These figures give the lengths in feet of one *second* of meridian and parallel respectively, at the latitudes stated.

From A to B it is clear that we go *north* about 18,000 ft. from the calculated difference of latitude, and this will correspond with about $\frac{18{,}000}{101{\cdot}5}$ seconds, or about 177 seconds, say 2′ 57″.

Hence the latitude of B will be *about* 54° 21′ 15″ + 2′ 57″, and the *mean* latitude will be, nearly enough, 54° 21′ 15″ + 1′ 30″, or 54° 22′ 45″.

Next we find the *approximate* difference of longitude. By interpolation, we find that at latitude 54° 22′ 45″, the length of one second of parallel is 59·2214 ft.

Hence the approximate difference of longitude in seconds

$$= \frac{\text{departure}}{\text{length of one second}} = \frac{45{,}355}{59{\cdot}2214} = 765{\cdot}9 \text{ seconds.}$$

Hence the convergence x = 765·9 × sin 54° 22′ 45″
= 622·6 seconds, say 10′ 23″.

This enables us to find the azimuth of BC at once.

		°	′	″
Bearing of AB	=	68	27	20
Included angle ABC	=	204	27	18
		272	54	38
		180	0	0
Bearing of BC	=	92	54	38
Convergence			10	23
Azimuth of BC	=	93	5	1

Convergence always *increases* the azimuth if we go *east*, but decreases it if we go *west*.

To find the latitude and longitude of B more exactly, we now recalculate difference of latitude and departure, but using the *mean* azimuth of AB, namely $68° \ 27' \ 20'' + 5' \ 11''$, or $68° \ 32' \ 31''$.

$$\text{Diff. of lat.} = 48{,}762 \cos 68° \ 32' \ 31'' = 17{,}838 \text{ ft.}$$
$$\text{Departure} = 48{,}762 \sin 68° \ 32' \ 31'' = 45{,}382 \text{ ft.}$$

At the mean latitude, the lengths of one degree of meridian and parallel are, by interpolation,

101·4469 and 59·2214 ft. respectively.

Hence the true differences of latitude and longitude in seconds respectively are $17{,}838 \div 101{\cdot}4469 = 175{\cdot}8$, and $45{,}382 \div 59{\cdot}2214 = 766{\cdot}3$. These are north and east, respectively.

Hence we have—

		° ′ ″				° ′ ″	
Latitude of A	=	54 21 15	N.	Longitude	=	2 21 14	W.
Correction	+	2 56			−	12 46	
Latitude of B	=	54 24 11	N.	Longitude	=	2 8 28	W.

These results are to the nearest second. If a more exact result is required, the convergence may be recalculated from the new difference of longitude, and the whole repeated.

It is clear that the known data at B are now the same as we had at A, and we can carry the calculation on to other stations as required.

CHAPTER VI

RIGHT-ANGLED SPHERICAL TRIANGLES

In Vol. I of this book we were able to draw all the projections dealt with by calculating distances along the meridians of longitude and parallels of latitude on the earth, assumed exactly spherical. There are, however, important projections in which it is necessary to calculate distances on the earth in other directions, and for this it is necessary to know a little spherical trigonometry. A slight acquaintance with this subject also enables us to realize the effect of the spherical shape of the earth on "Geodetic Surveying" (i.e. the survey of large regions), and to solve some other interesting problems relevant to geography.

Great and Small Circles.

Any section of a sphere by a plane is a circle; if the plane passes through the centre of the sphere the circle is called a "great circle," and its radius is that of the sphere. All others are called "small circles." Thus meridians of longitude and the equator are great circles, while parallels of latitude are small circles. A great circle is the shortest distance between two points lying on it, and so corresponds to a straight line on a plane. When we move in a "straight line" on the earth's surface we are really moving along a great circle, but with this difference: on a plane the *direction* of a straight line is always the same, but on a sphere the direction (or "azimuth") of a great circle is constantly changing unless it is a meridian (when the direction is north or south) or the equator (when the direction is east or west). At all other latitudes it is impossible to move east or west in a "straight line" for any great distance, as the parallels of latitude are small circles and, therefore, appear to us as *curves*, not straight lines.

Spherical Triangles.

A spherical triangle is the figure formed on the surface of a sphere by arcs of three great circles, which are called its

"sides." The lengths of the sides are proportional to the angles they subtend at the centre of the sphere, as the length of a side = radius of sphere × circular measure of the angle subtended at the centre. As the radius of the sphere is constant we can measure any side by the *angle* it subtends at the centre of the sphere. The three "angles" of the triangle are the angles contained by the tangents to the sides of the triangle at the intersections of the great circles, i.e. the angles between the planes of the great circles. The formulae in spherical trigonometry are thus all relations between these latter "angles" and the former angles which are the measures of the "sides."

Spherical Excess.

The great difference between plane and spherical trigonometry is that in the former the sum of the three angles is always two right angles, or 180° or π radians: in the latter it is always greater than 180° by an amount called the "spherical excess," which can be shown to be $\frac{\text{area of triangle}}{(\text{radius of sphere})^2}$ in circular measure. For example, the triangle contained by the equator and the meridians of 0° and 90° longitude obviously contains three right angles or the spherical excess is one right angle.

This agrees with the above general rule according to which the excess should be $\frac{\frac{1}{8}\text{ area of sphere}}{(\text{radius of sphere})^2} = \frac{\frac{1}{8}\,4\pi R^2}{R^2} = \frac{\pi}{2}$ = one right angle. As the earth is nearly a sphere of 3,957 miles radius, and as one radian = 206,265″, the spherical excess in *seconds* $= \frac{\text{area of triangle in sq. miles}}{(3957)^2} \times 206{,}265$ $= \frac{\text{area in sq. miles}}{75{\cdot}9}$. The excess is clearly only important in large triangles, e.g. if the sides are each 100 miles long the area is 4,330 sq. miles, and the sum of the three angles will be 180° 0′ 57″, but in a spherical triangle of any size we cannot, if we know two of its angles, find the third angle by merely deducting the sum of the two angles from 180°

Right-angled Triangles.

It will be sufficient for the purposes of this volume to deal fully only with right-angled triangles, i.e. spherical triangles in which *one* of the angles is a right angle. A triangle may have two right angles, e.g. the triangle formed by the equator and any two meridians, in which case the remaining angle is equal to the side between the two right angles and the other two sides each equal 90°. As mentioned above, a triangle may have three right angles, in which case each of the sides is 90°. Let ABC (Fig. 43) be a spherical triangle, in which C is a right angle, and let O be the centre of the sphere.

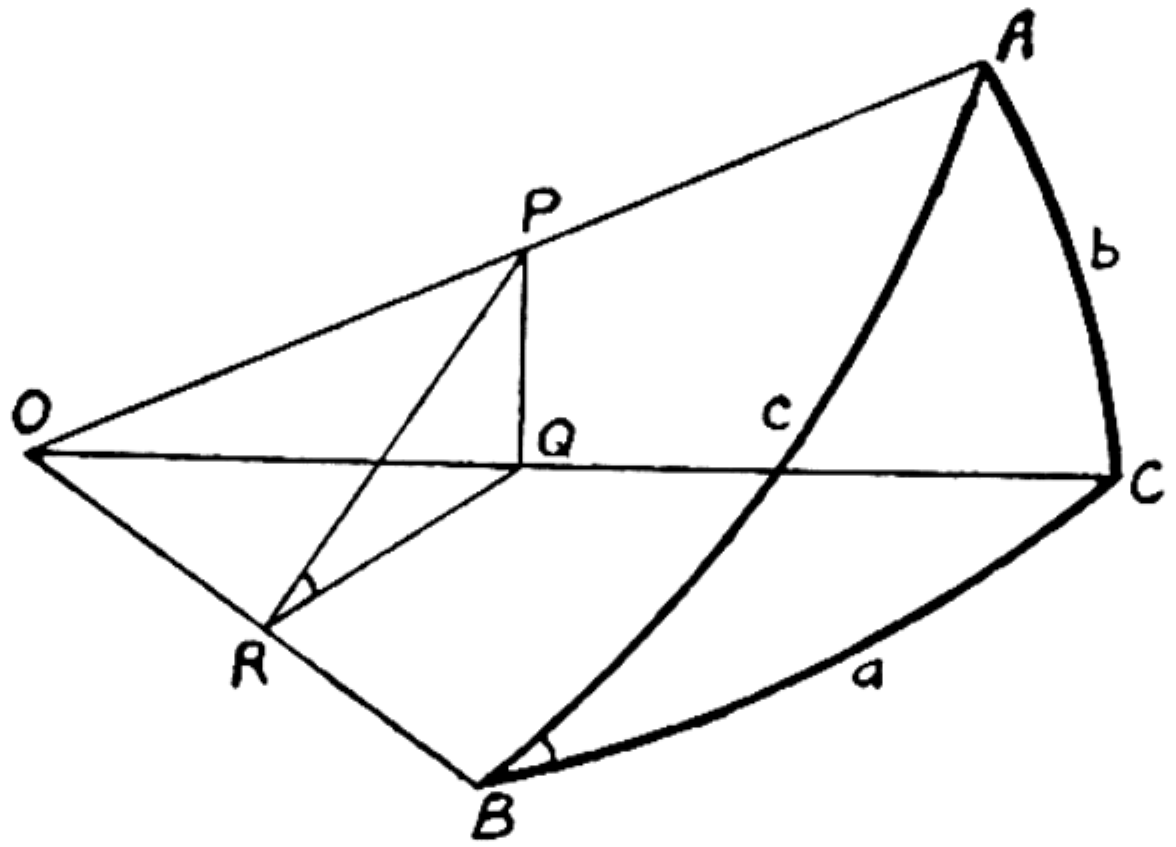


FIG. 43

Join OA, OB, OC. Let us call the sides $AB = c = \angle AOB$, $BC = a = \angle BOC$, $CA = b = \angle COA$ From P any point in OA draw $PQ \perp OC$, draw $QR \perp OB$ and join PR. By construction PQO, QRO are right angles and PQR is also a right angle for the planes AOC, COB are at right angles, and PQ is drawn $\perp$ to their intersection OC—it is therefore $\perp$ to the plane COB and, therefore, to all lines in it. Also $OP^2 = PQ^2 + OQ^2 = PQ^2 + QR^2 + OR^2 = PR^2 + OR^2$. $\therefore \angle PRO$ is a right angle. In fact the "tetrahedron" $OPQR$ is composed of four right-angled triangles. Also since PR is parallel to the tangent to side AB at B, and QR is parallel to the side BC at B, the angle PRQ = angle B of the spherical triangle, while $\angle POR = c$, $\angle QOR = a$, $\angle POQ = b$.

$$\text{Then we have } \frac{OR}{OP} = \frac{OR \cdot OQ}{OQ \cdot OP} \quad \therefore \cos c = \cos a, \cos b \qquad (1)$$

$$\text{Also } \sin B = \frac{PQ}{PR} = \frac{PQ \cdot OP}{OP \cdot PR} \quad \therefore \sin B = \frac{\sin b}{\sin c} \quad . \qquad (2)$$

$$\cos B = \frac{QR}{PR} = \frac{QR \cdot OR}{OR \cdot PR} \quad \therefore \cos B = \frac{\tan a}{\tan c} \quad . \qquad (3)$$

$$\tan B = \frac{PQ}{QR} = \frac{PQ \cdot OQ}{OQ \cdot QR} \quad \therefore \tan B = \frac{\tan b}{\sin a} \quad . \quad (4)$$

It is obvious that, starting from a point in OB instead of OA, we can prove formulae similar to formulae (2), (3) and (4) for angle A, which will be as follows—

$$\sin A = \frac{\sin a}{\sin c} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\cos A = \frac{\tan b}{\tan c} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$\text{and } \tan A = \frac{\tan a}{\sin b} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Then from (4) and (7)—

$$\tan A, \tan B = \frac{\tan a}{\sin b} \cdot \frac{\tan b}{\sin a} = \frac{1}{\cos a \cdot \cos b} = \frac{1}{\cos c}$$

$$\therefore \cos c = \cot A \cdot \cot B \quad . \quad . \quad . \quad . \quad (8)$$

and from (3) and (5)—

$$\frac{\cos B}{\sin A} = \frac{\tan a}{\tan c} \cdot \frac{\sin c}{\sin a} = \frac{\cos c}{\cos a} = \cos b.$$

$$\therefore \cos B = \sin A \cdot \cos b \quad . \quad . \quad . \quad . \quad (9)$$

and from (2) and (6)—

$$\frac{\cos A}{\sin B} = \frac{\tan b}{\tan c} \cdot \frac{\sin c}{\sin b} = \frac{\cos c}{\cos b} = \cos a.$$

$$\therefore \cos A = \sin B \cdot \cos a \quad . \quad . \quad . \quad . \quad (10)$$

These ten formulae are all very simple and easy to use with tables of the logarithms of sines, cosines and tangents, but they are confusing to remember. There is, however, an easy way of writing down the appropriate formula when we are given any of two of the *five* parts (three sides and two angles, the right angle being ignored for this purpose), and wish to find any of the other three. This mnemonic is known as

"Napier's Five-part Circle," as it was devised by John Napier, of Murchiston (1550–1617), the inventor of logarithms, and is as follows: Sketch a circle (Fig. 44) and divide it into 5 "parts" (sectors); in these write down *in order* the two sides b and a adjacent to the right angle C and the complements of the other three parts, i.e. $90° - B$, $90° - c$, $90° - A$. Then, considering any part of the circle as a "middle part," we have

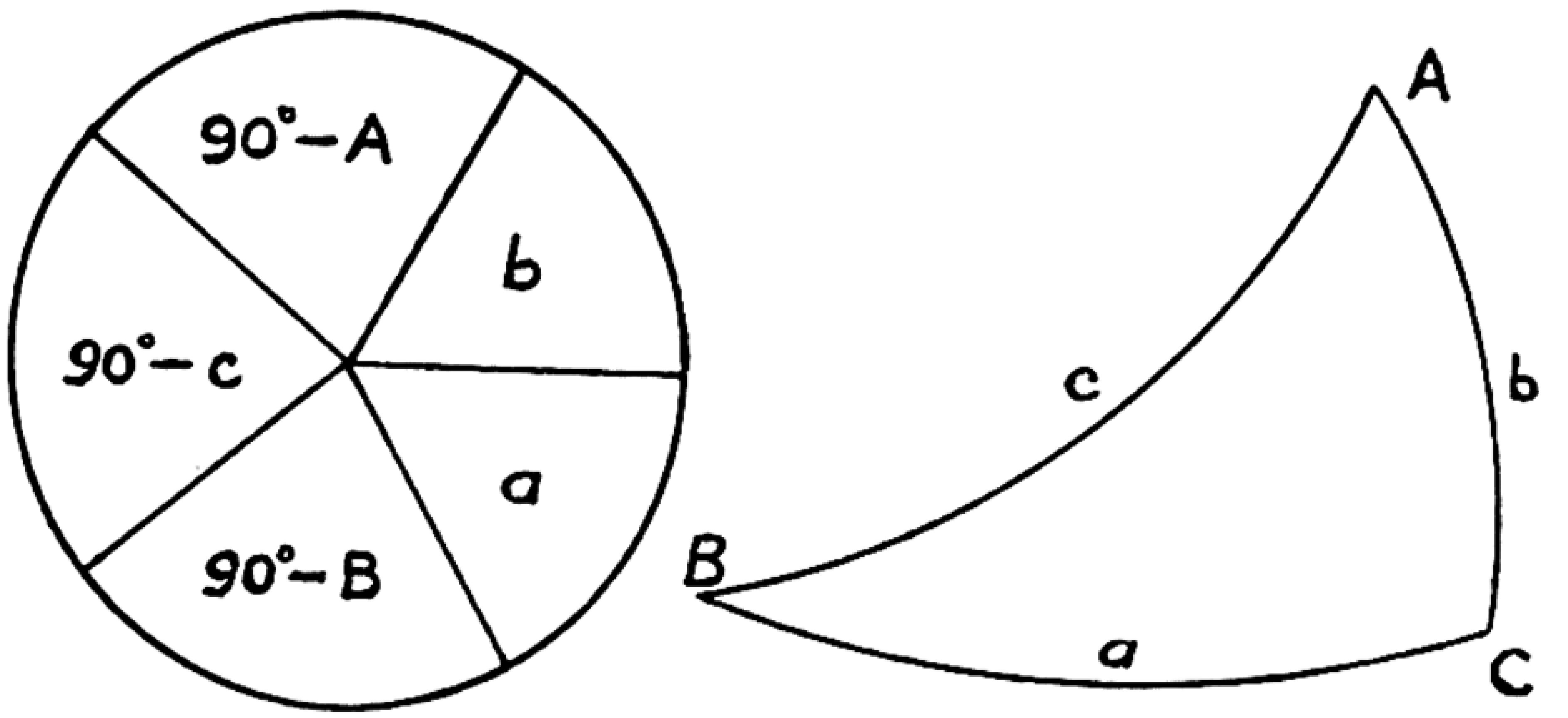


FIG. 44

two parts "adjacent" to it and two parts "opposite" to it. The rule now is—

The *sine* of a *middle* part
= product of *tangents* of *adjacent* parts
= product of *cosines* of *opposite* parts.

Notice that "sine" and "middle," "tangents" and "adjacent," "cosines" and "opposite" have in all cases similar vowels. As we can take any of the five parts as "middle part" and write down two formulae from it, we can write down ten formulae which will be found to agree with those proved above, e.g. choosing $90° - c$ as middle part, we have $\sin(90° - c) = \tan(90° - A), \tan(90° - B) = \cos a \cdot \cos b$, or, $\cos c = \cot A \cdot \cot B = \cos a \cdot \cos b$ as in formulae (8) and (1) above. The student should test the method to obtain the other eight formulae by taking in turn each of the other four parts as middle part and writing down the corresponding formulae. As illustrations of the application of these formulae to geographical problems, we shall now give a few numerical examples.

Example 1.

To find the distance of Bombay (18° 55′ *N.*, 72° 54′ *E.*) *from London* (51° 31′ *N.*, 0° 6′ *W.*), *and the direction of Bombay from London, taking the earth as a sphere of radius* 3,957 *miles.* This illustrates the general problem of finding the

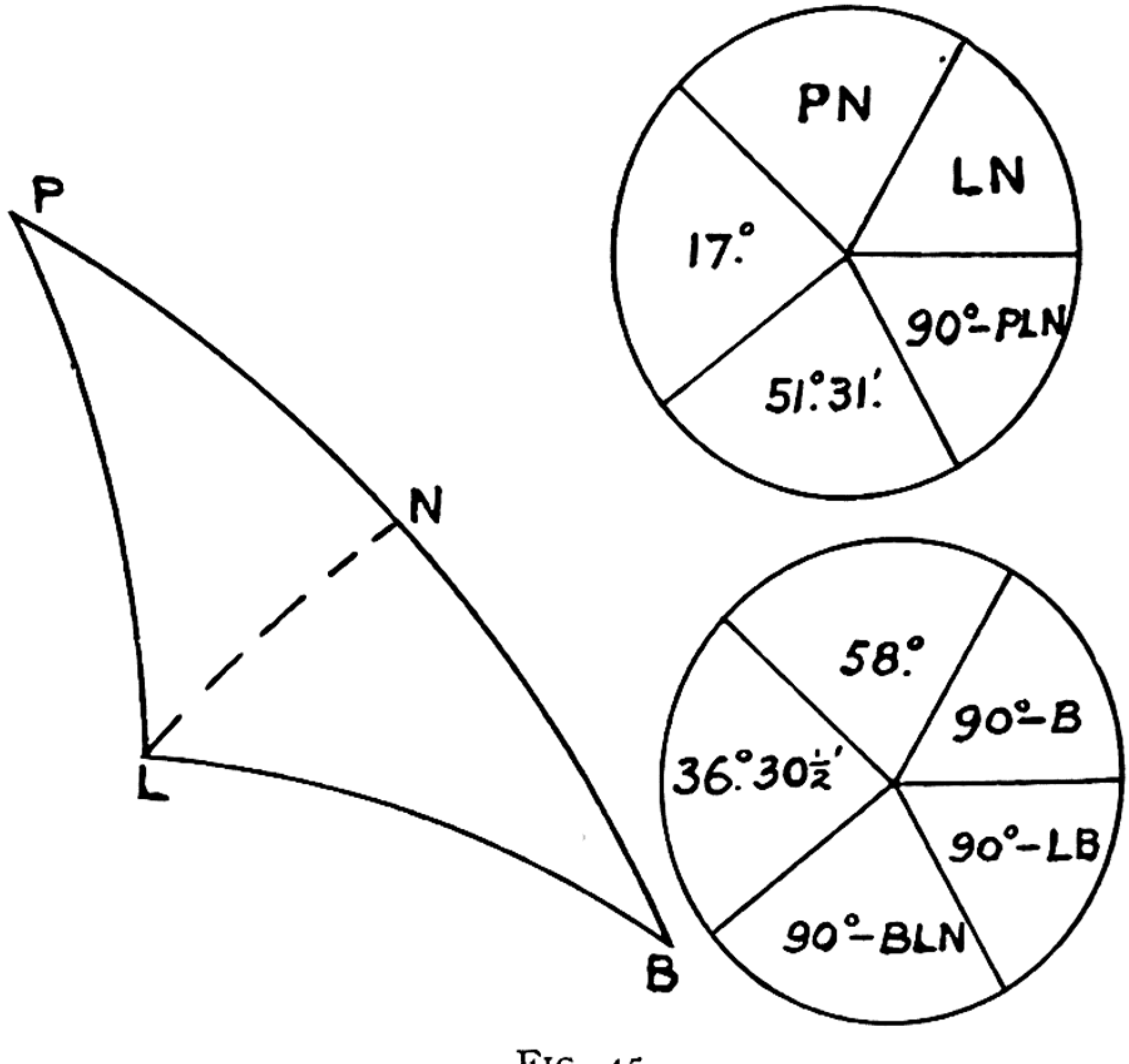


FIG. 45

distance and azimuth of any point on the earth's surface from any other point.

In Fig. 45 let L be London, B Bombay, and P the North Pole, then PLB is an oblique spherical triangle, of which the sides, $PL = 90° - 51°\ 31' = 38°\ 29'$, $PB = 90° - 18°\ 55' = 71°\ 5'$, and the angle $P = 0°\ 6' + 72°\ 54' = 73°\ 0'$ are known. If we imagine a great circle $LN \perp PB$, we divide PLB into two right-angled spherical triangles, PLN and BLN, and we sketch the five-part circles for these.

Then in triangle PLN we have $\sin 17° = \tan PN \cdot \tan 51°\ 31'$. $\therefore \tan PN = \dfrac{\sin 17°}{\tan 51°\ 31'}$. $\therefore \log \tan PN = \bar{1}{\cdot}4659$

$- \cdot 0997 = \bar{1} \cdot 3662$. $\therefore PN = 13^\circ\ 5'$. $\therefore BN = 71^\circ\ 5' - 13^\circ\ 5' = 58^\circ\ 0'$. Also $\sin LN = \cos 17^\circ \cdot \cos 51^\circ\ 31'$. $\therefore \log \sin LN = \bar{1} \cdot 9806 + \bar{1} \cdot 7939 = \bar{1} \cdot 7745$. $\therefore LN = 36^\circ\ 30\frac{1}{2}'$. Also $\sin 51^\circ\ 31' = \tan 17^\circ \cdot \cot PLN$. $\therefore \log \tan PLN = \bar{1} \cdot 4853 - \bar{1} \cdot 8936 = \bar{1} \cdot 5917$. $\therefore PLN = 21^\circ\ 20\frac{1}{4}'$.
Then in triangle BLN, $\sin (90^\circ - LB) = \cos BN \cdot \cos LN$. $\therefore \cos LB = \cos 58^\circ\ 0' \cdot \cos 36^\circ\ 30\frac{1}{2}'$.

Hence $LB = 64^\circ\ 47\frac{1}{3}' = 1 \cdot 1308$ in circular measure. $\therefore$ the distance LB from London to Bombay $= 1 \cdot 1308 \times 3{,}957$ miles $= 4{,}475$ miles.

Again, $\sin LN = \tan (90^\circ - BLN)$, $\tan BN$. $\therefore \tan BLN = \dfrac{\tan 58^\circ\ 0'}{\sin 36^\circ\ 30\frac{1}{2}'}$. $\therefore BLN = 69^\circ\ 36\frac{1}{4}'$. $\therefore$ angle $PLB = 21^\circ\ 20\frac{1}{4}' + 69^\circ\ 36\frac{1}{4}' = 90^\circ\ 56\frac{1}{2}'$ = azimuth of line LB at L = direction of Bombay from London.

Also $\sin BN = \tan (90^\circ - B)$. $\tan LN$. $\therefore \tan B = \dfrac{\tan 36^\circ\ 30\frac{1}{2}'}{\sin 58^\circ\ 0'}$. $\therefore B = 41^\circ\ 7'$.

$\therefore$ London bears $41^\circ\ 7'$ west of north from Bombay, or the "reverse azimuth" of line $LB = 318^\circ\ 53'$.

With a greater knowledge of spherical trigonometry, the problem can be solved more directly, and in map projection tables of (angular) distance and azimuth can be used.

Example 2.

To find the latitude and longitude of intermediate points on the great circle from London to Bombay so as to enable the great circle to be plotted on a map on any projection.

In Fig. 46 let LM, BN be the meridians of London and Bombay respectively, MN being the equator and let the great circle LB be produced to meet MN at A. Let X be any intermediate point between L and B, XY its meridian, ϕ its latitude $= XY$ and x the distance of Y from A along the equator. Then LMA, BNA and XYA are all right-angled triangles, $LM = 51^\circ\ 31'$, $BN = 18^\circ\ 55'$, $MN = 73^\circ\ 0'$.

The five-part circle for XYA will serve for the other two triangles, "mutatis mutandis." Then $\sin x = \dfrac{\tan \phi}{\tan A}$.

Similarly $\sin MA = \dfrac{\tan 51^\circ\ 31'}{\tan A}$, $\sin NA = \dfrac{\tan 18^\circ\ 55'}{\tan A}$.

$$\therefore \frac{\sin MA}{\sin NA} = \frac{\tan 51^\circ\ 31'}{\tan 18^\circ\ 55'} = 3{\cdot}671.$$

Also $MA = 73^\circ + NA$. $\therefore \sin MA = \sin 73^\circ \cos NA + \cos 73^\circ \sin NA$.

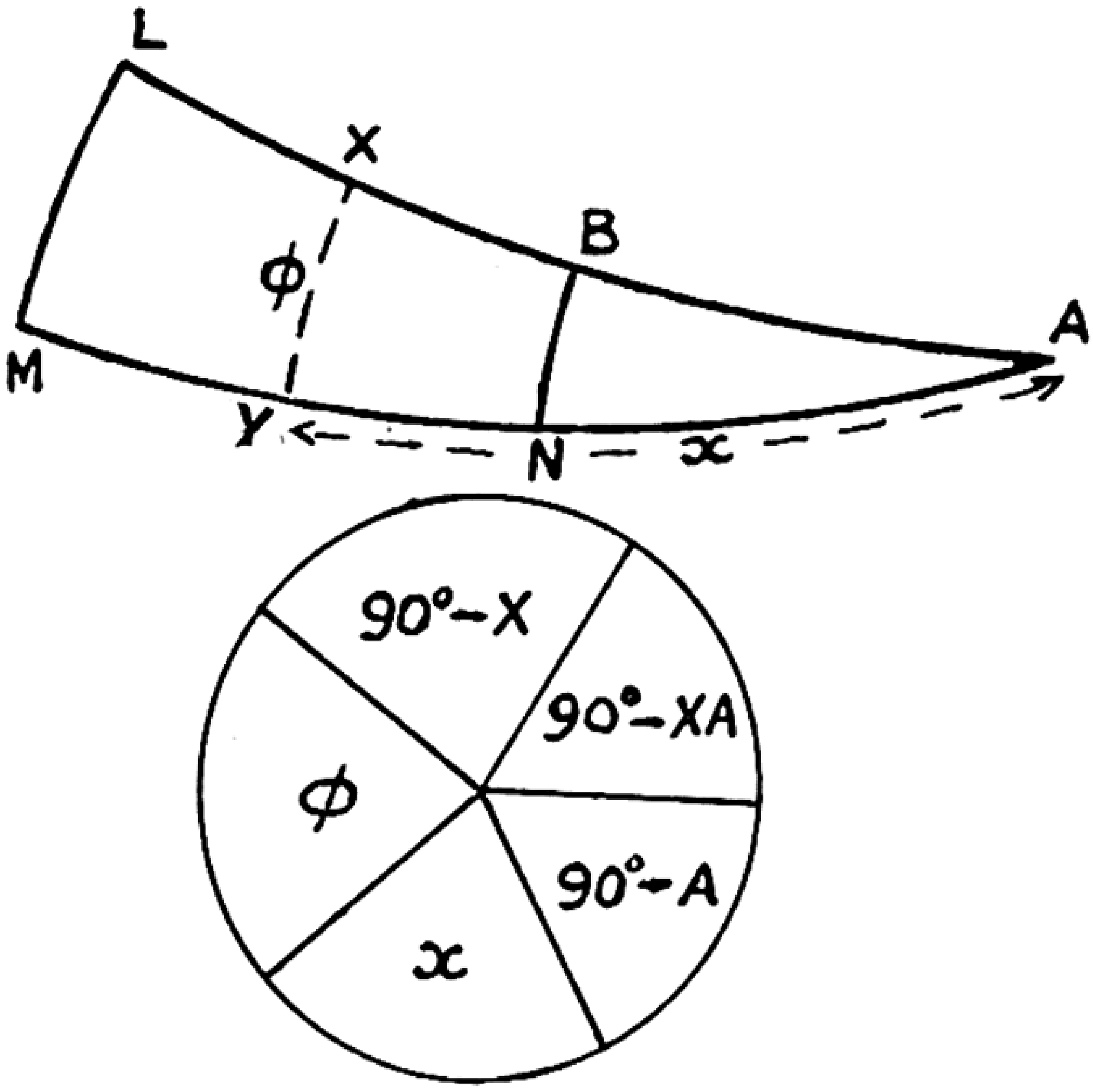


FIG. 46

$$\therefore \frac{\sin MA}{\sin NA} = \sin 73^\circ \cot NA + \cos 73^\circ = 3{\cdot}671 \text{ (above).}$$

$\therefore \cot NA = \dfrac{3{\cdot}671 - {\cdot}2924}{{\cdot}9563} = 3{\cdot}533$. $\therefore NA = 15^\circ\ 48'$.

$\therefore$ longitude of $A = 72^\circ\ 54' + 15^\circ\ 48' = 88^\circ\ 42'$ E. and $MA = 88^\circ\ 48'$.

Also $\tan A = \dfrac{\tan 51^\circ\ 31'}{\sin 88^\circ\ 48'}$ (above). $\therefore A = 51^\circ\ 31\tfrac{1}{2}'$. $\therefore \tan \phi = \sin x \cdot \tan A$, $\therefore \log \tan \phi = \log \sin x + {\cdot}0998$.

We can now prepare a table showing the latitude at which each 10° meridian of longitude will be crossed. For the 10° meridian x will be $78^\circ\ 42'$, etc.

Longitude	x	log sin x	log tan ϕ	ϕ = latitude
10° E.	78° 42′	$\bar{1}$·9915	0·0913	50° 59′ N.
20° E.	68° 42′	$\bar{1}$·9693	0·0691	49° 32′ N.
30° E.	58° 42′	$\bar{1}$·9317	0·0315	47° 05′ N.
40° E.	48° 42′	$\bar{1}$·8758	$\bar{1}$·9756	43° 23′ N.
50° E.	38° 42′	$\bar{1}$·7960	$\bar{1}$·8958	38° 11′ N.
60° E.	28° 42′	$\bar{1}$·6814	$\bar{1}$·7812	31° 8′ N.
70° E.	18° 42′	$\bar{1}$·5060	$\bar{1}$·6058	21° 58′ N.

The student should mark these points on, say, Mercator's Projection, and join them with a fair curve as an example of a great circle course for an airship.

Example 3.

Convergence of Meridians in Surveying. This has already been referred to on page 65. If A and B (Fig. 47) are two points at the same latitude (ϕ) and P is the pole, PA and PB the meridians through A and B are each equal $90° - \phi$, and $\angle APB = \theta =$ difference of longitude of B and A. Let ACB be the great circle through A and B, which we call the "straight line" AB in surveying, while the curve $AC'B$ is the parallel of latitude, ϕ. Let the azimuth or bearing of line AB at A be α, and let its azimuth at B be $\alpha + x$. From symmetry we can see that $\alpha = \angle PAB = \angle PBA = 180° - \alpha - x$, or $\alpha = 90° - \frac{x}{2}$, while $\angle PBE = \alpha + x = 90° + \frac{x}{2}$. If the point C bisects AB and we draw the great circle PC, by symmetry $\angle PCA = \angle PCB = 90°$, and $\angle APC = \angle BPC = \frac{\theta}{2}$. $\therefore$ PCA is a right-angled spherical triangle.

From the five-part circle we have $\sin\phi = \tan\left(90° - \frac{\theta}{2}\right)$, $\tan\frac{x}{2}$. $\therefore \tan\frac{x}{2} = \tan\frac{\theta}{2}\cdot\sin\phi$. Now for such "straight lines" as we can range in surveying, the difference of longitude, θ, is only a small angle, $\therefore \frac{\theta}{2}$ is small and so also will be $\frac{x}{2}$. Thus we can write the circular measure instead of the tangent and then write x and θ in any units, e.g. seconds.

$\therefore x = \theta \cdot \sin \phi$. It can be shown that if the line is in any direction so that A and B are not necessarily at the same latitude, the formula is still true for such short distances as we are considering if for ϕ we substitute $\overline{\phi}$, the average latitude of A and B, so that we have the simple formula— $x = \theta \cdot \sin \overline{\phi}$, or "*the increase of azimuth along a straight line*

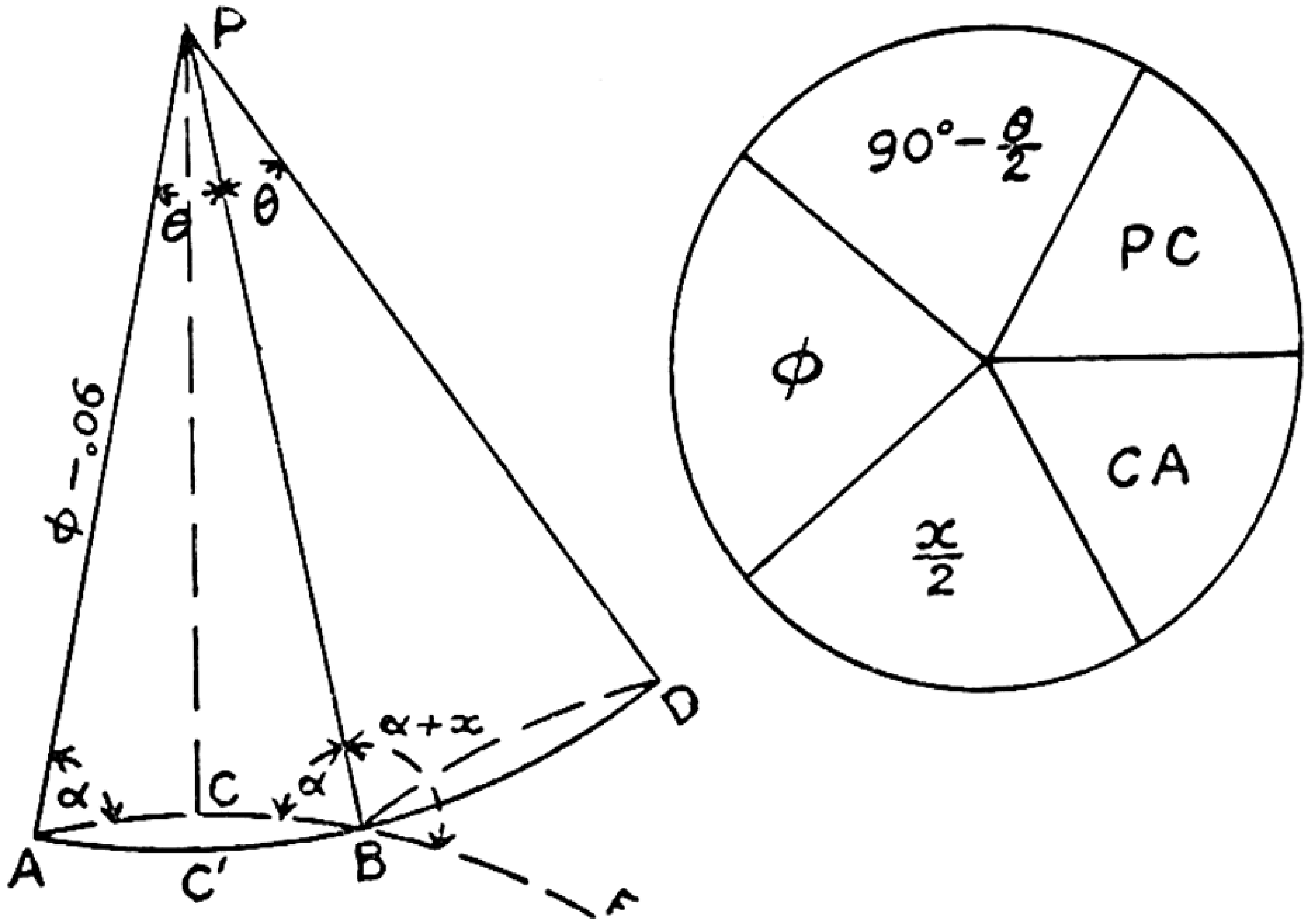


FIG. 47

= difference of longitude of its ends × sine of average latitude of its ends."

For example, if a straight line 1 mile long has its ends at latitude 51° 30′, the difference of longitude of its ends will be 5,280 ÷ length of 1″ longitude at latitude 51° 30′ $= \frac{5280}{63 \cdot 3}$ $= 83 \cdot 4''$, so the increase in azimuth will be $83 \cdot 4'' \times \sin 51° \, 30'$ $= 83 \cdot 4'' \times \cdot 7826 = 65 \cdot 3'' = 1' \, 5''$, an amount which is quite appreciable even in such a short line.

We could use this formula to set out the parallel of latitude, 51° 30′, on the earth's surface as for an international boundary, thus: At A find the latitude, 51° 30′, and the meridian, AP, by astronomical observations, and from AP turn off an angle $90° - \frac{x}{2} = 90° - 32 \cdot 6''$, and range a line AB in this direction

for a length of 5,280 ft., then at B deflect through an angle $x = 65{\cdot}3''$ to the left $\left(= 90° + \frac{x}{2} - \left(90° - \frac{x}{2}\right)\right)$, and range another line BD of 5,280 ft., and so on—in fact we should be setting out a curve of very large radius in chords of 1 mile.

Example 4.

Latitude and Longitude from Survey Measurements as in a Triangulation. See Numerical Example, page 87. It would be very laborious to determine the latitude and longitude of every station in a triangulation by astronomical methods, but if we know them for one end, A, of a line AB (Fig. 48) we can deduce them for the other end B, knowing the length, l, of the line and its azimuth α at A. We first find the latitude and longitude of B to a first approximation thus difference of latitude

$$\delta\phi = \frac{l \cos \alpha}{\text{length of } 1'' \text{ of latitude at } A},$$ then the average latitude $\bar{\phi}$ is latitude of $A = \frac{\delta\phi}{2}$.

Difference of longitude

$$= \frac{l \sin \alpha}{\text{length of } 1'' \text{ of latitude at average latitude, } \bar{\phi}}$$

$= \delta\theta$. Then, the increase of azimuth $= x = \delta\theta \cdot \sin \bar{\phi}$. Then we recalculate $\delta\phi$ and $\delta\theta$, substituting for α the average value of the azimuth of the line AB, viz., $\alpha + \frac{x}{2}$, and thus find the latitude and longitude of B to a second approximation, and so on to any desired accuracy. This method also enables us to *project* all the triangulation stations on to the same meridian, when measuring an arc of a meridian in order to determine the shape of the spheroid.

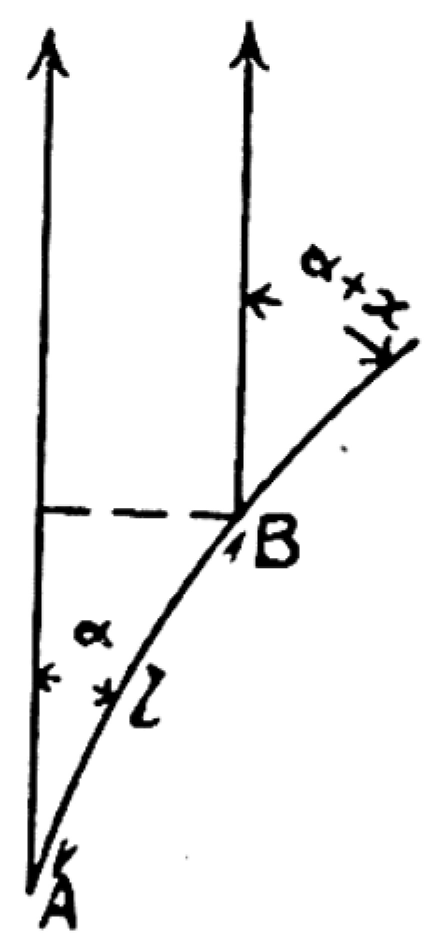
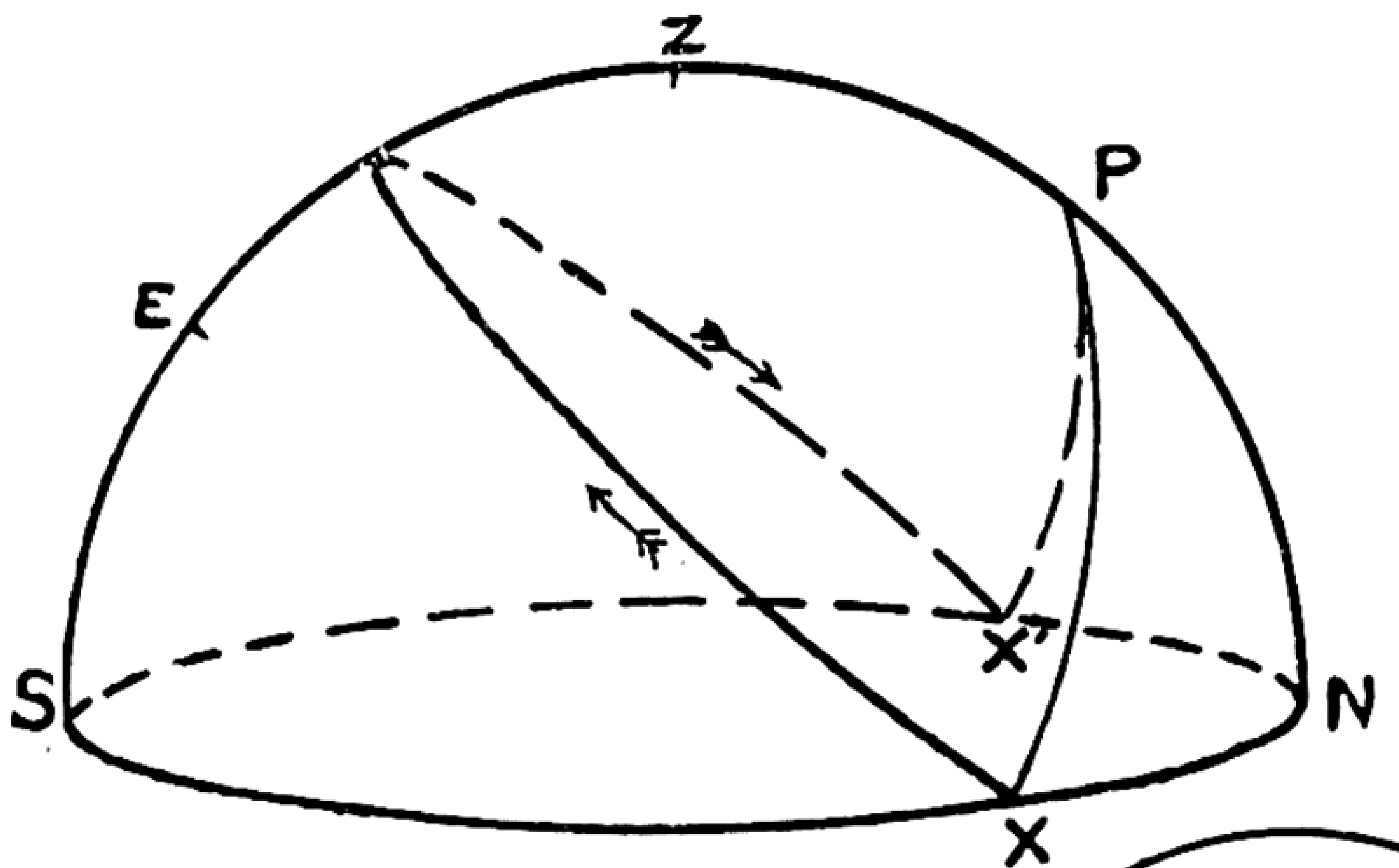


FIG. 48

Example 5.

To Find the Times of Sunrise and Sunset at any place—say London, latitude 51° 30′, *on any date, say* 21*st June.* In Fig. 49, P is the celestial pole, NPS is the meridian, NXS is the horizon, X is the sun's centre at sunrise, X' its centre at

sunset. Then $PN = 51^\circ\ 30'$, $PX = PX' = 90^\circ - 23^\circ\ 27' = 66^\circ\ 33'$, as the sun's declination on 21st June is $23^\circ\ 27'$, and is practically constant for that day. Then PNX is a right-angled spherical triangle with $\angle N = 90^\circ$, and we have to find the angle P. From the five-part circle we have $\sin(90^\circ - P) = \tan 23^\circ\ 27', \tan 51^\circ\ 30'$. $\therefore \log\cos P = \bar{1}{\cdot}6372 + 0{\cdot}0994$



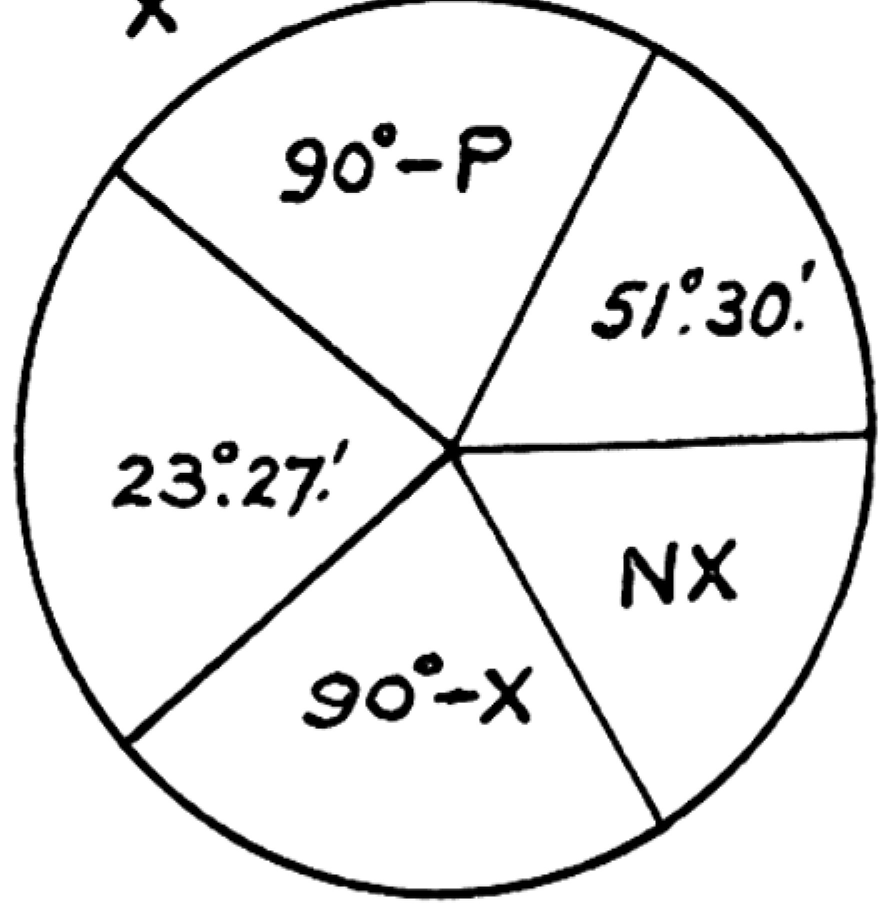


Fig. 49

$= \bar{1}{\cdot}7366$. $\therefore P = 56^\circ\ 57' = 3$ h. 47 m. 48 s. Therefore, the "local apparent time" of sunrise is 3 h. 47 m. 48 s. a.m., and of sunset 8 hr. 12 m. 12 s. p.m.

The equation of time is + 1 m. 33 s. on 21st June at noon, increasing 13 s. per day: it is, therefore, 1 m. 29 s. at sunrise and 1 m. 37 s. at sunset. The local mean times of sunrise and sunset are, therefore, 3 h. 47 m. 48 s. + 1 m. 29 s. = 3 h. 49 m. 17 s. a.m., and 8 h. 12 m. 12 s. + 1 m. 37 s. = 8 h. 13 m. 49 s. p.m.—say 3 h. 49 m. a.m. and 8 h. 14 m. p.m.

Actually the times given in *Whitaker's Almanac* are 3 h. 45 m. a.m. and 8 h. 19 m. p.m. as refraction enables us to see the sun when it is just wholly below the horizon. This correction for refraction is greater the less the inclination of the sun's path to the horizon, i.e. the greater the latitude, and becomes much larger at high latitudes.

Example 6.

To Find the Azimuth of a Survey Line by Observing the Elongation of a Circumpolar Star. When the declination of a star is greater than the latitude of the place of observation, the star at its upper culmination passes between the zenith and the celestial pole, as EZ in Fig. 50 equals the latitude. If we observe such a star with a theodolite (in the northern hemisphere) it will appear to travel from north (lower culmination) eastward, then back to north again (upper culmination) then westward, then back to north again. As it approaches its most easterly and westerly positions it appears to travel more and more slowly in a horizontal direction, then to stop, then to travel back again just as a man running on a circular track would appear to a spectator on a stand outside the track. At its most easterly and westerly positions the star is said to "elongate," and it appears to travel up or down the vertical hair of the theodolite for a few minutes, affording time to read the horizontal circle, change face, and sight again on the star without any appreciable change in the azimuth of the star during the process if the star is near the pole. At the elongations S_1, S_2 the angle ZSP is a right-angle, so that we have $\sin SP = \cos(90° - Z)$, $\cos(90° - ZP)$, or cos declination $= \sin Z \times$ cos latitude. $\therefore \sin Z = \dfrac{\text{cos declination}}{\text{cos latitude}}$, also $\sin(90° - P) = \tan SP \cdot \tan(90° - ZP)$, or $\cos P = \dfrac{\text{tan latitude}}{\text{tan declination}}$.

If ZA is the survey line whose azimuth is required, the order of operations is as follows: Sight on A, reading the horizontal circle, then sight on the star until it appears to begin to travel up or down the vertical hair, read the horizontal circle, change face, sight on the star again, read the horizontal circle, then resight on A and read the horizontal circle. The average of the two values of the horizontal angle AZS should then be free

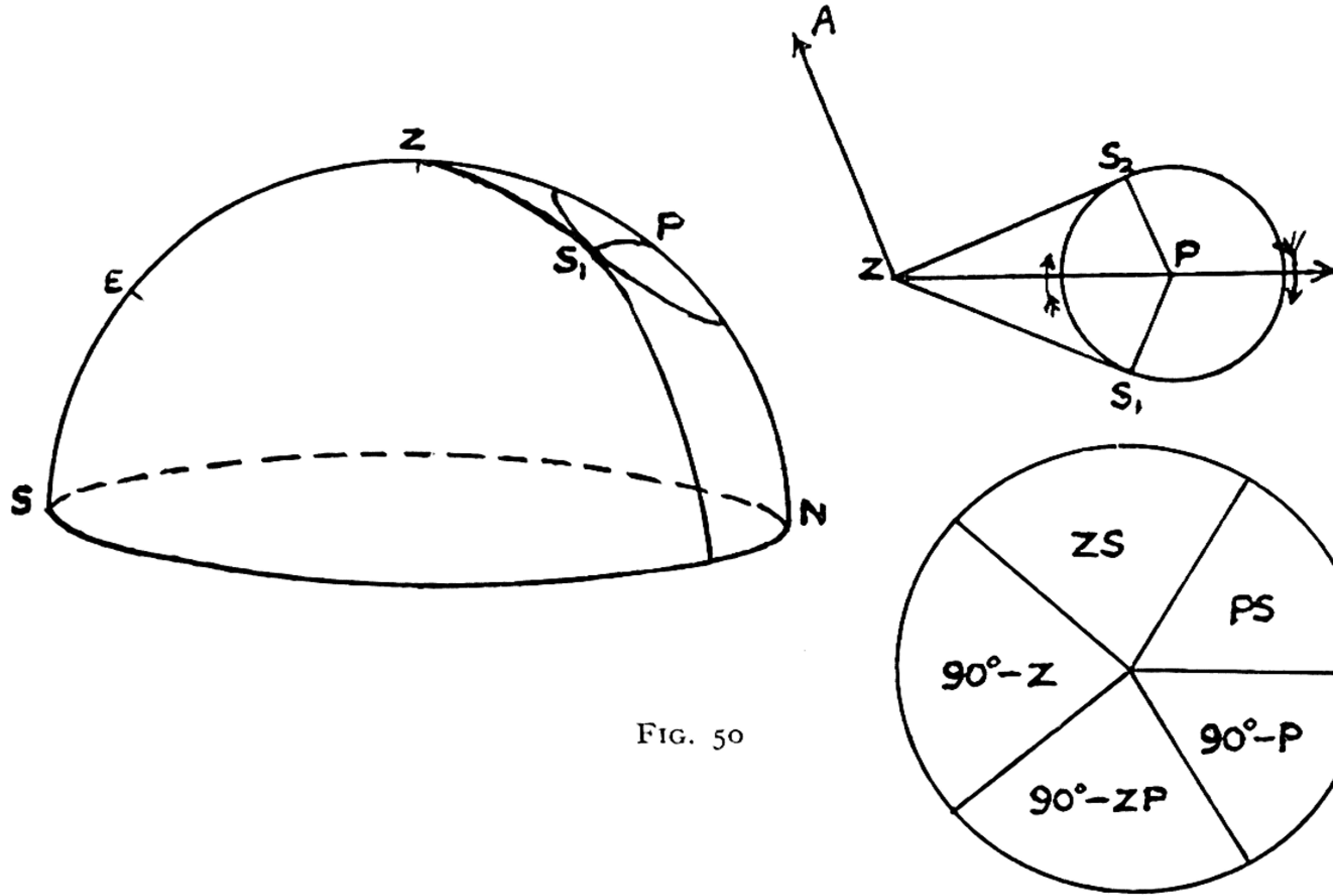


FIG. 50

from instrumental error. By subtracting the calculated angle PZS_1, or adding the calculated angle PZS_2 to the angle AZS in Fig. 50, we find the angle AZP and hence the azimuth of line AZ.

As a numerical example: the Pole Star, declination 88° 55′ N., R.A. 1 h. 35 m. is observed at westerly elongation in latitude 51° 30′ N., when its clockwise bearing from a survey line ZA is 84° 20′. Find the azimuth of line ZA and the L.S.T. of the observation.

We have $\sin Z = \dfrac{\cos 88^\circ\ 55'}{\cos 51^\circ\ 30'}$. $\therefore \angle PZS_2 = 1^\circ\ 44'$. $\therefore AZP = 84^\circ\ 20' + 1^\circ\ 44' = 86^\circ\ 04'$. $\therefore ZA$ bears 86° 04′ west of north or 273° 56′ east of north.

Also $\cos P = \dfrac{\tan 51^\circ\ 30'}{\tan 88^\circ\ 55'}$. $\therefore$ Hour angle $ZPS_2 = 88^\circ\ 38'$ = 5 h. 54 m. 32 s. $\therefore$ L.S.T. = 5 h. 54 m. 32 s. + 1 h. 35 m. = 7 h. 29 m. 32 s. from which the L.M.T. of the elongation can be ascertained *beforehand* in order to be ready just before the elongation occurs.

(*N.B.* The R.A. of the Pole Star varies rather considerably, but is given at 20-day intervals in *Whitaker's Almanack*.)

CHAPTER VII

MORE ADVANCED CYLINDRICAL AND ZENITHAL PROJECTIONS

Cassini's Projection.

As stated in Vol. I, this is the transverse simple cylindrical, the cylinder touching the reduced earth along the central meridian of the region instead of along the equator. We must imagine great circles at right angles to the central meridian passing through each point of the region ; the distance from the central meridian to any point measured along such a great circle will then form one co-ordinate, x, of the point on the map, while the distance along the central meridian from the equator (say) to the point where this great circle meets it will give the other co-ordinate, y. We then plot this y and x along and at right angles to the central meridian for each intersection of meridians and parallels, and join the points so found with curved lines to give the net of meridians and parallels for our map.

The defect will, of course, be that our x co-ordinates on the map will be measured parallel to each other, while on the earth they are only parallel at the central meridian and converge towards one another more and more as they depart from it, and would ultimately all meet at a point called the "pole" of the central meridian, 90° from it along each great circle. The result is an exaggeration of the scale parallel to the central meridian equal to the secant of the angular distance of the point from the central meridian along its great circle, but if this angle is not greater than 2° 34′ the exaggeration is not greater than $\frac{1}{1000}$th part. This angle at the earth's centre subtends a distance of about 177 miles on the earth's surface, so for a country which does not extend much more than this distance from its central meridian the projection is a good one, and it is thus used for the British Ordnance maps, $\frac{1}{2500}$ and 6 in. and 1 in. to the mile.

Let A (Fig. 51) be a point which is at the intersection of the meridian at longitude θ from the central meridian with the

parallel of latitude ϕ. If P is the pole and AN the great circle drawn $\perp$ to the central meridian, PAN is a right-angled spherical triangle, where $\angle P = \theta$, $PN = 90° - y$, $PA = 90° - \phi$, and $AN = x$ Then $sin\ x = sin\ \theta \,.\, cos\ \phi$. Also, $\cos \theta$

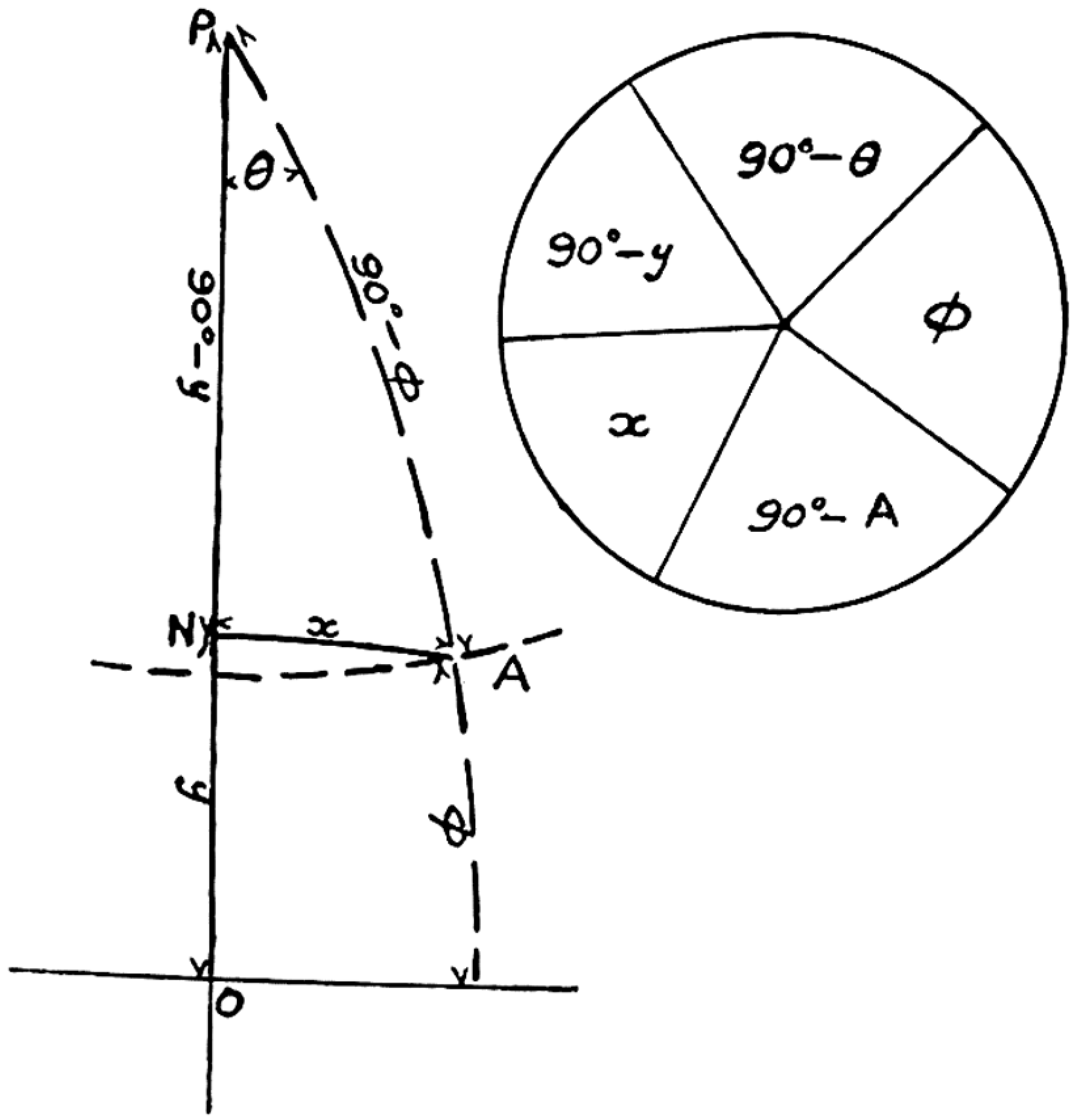


FIG. 51

$= \tan \phi \,.\, \cot y$. $\therefore\ tan\ y = \dfrac{tan\ \phi}{cos\ \theta}$. These formulae give the co-ordinates x and y as *angles* subtended at the earth's centre.

To draw Cassini's projection for Great Britain, 50° – 58° N., 6° W – 2° E., taking 2° W. as our central meridian, and with meridians and parallels at 2° intervals; points on the central meridian itself will be at true distances apart as when $\theta = 0°$, $\cos \theta = 1$, and $y = \phi$. Work with 4-figure logarithms in columns headed, θ, ϕ, log sin θ, log cos ϕ, log sin x, x; log tan ϕ, log cos θ, log y, y.

We thus obtain the following table—

θ	ϕ	x	y	θ	ϕ	x	y
2°	50° 52° 54° 56° 58°	1° 17′ 1° 14′ 1° 10½′ 1° 07′ 1° 04′	50° 01′ 52° 01′ 54° 01′ 56° 01′ 58° 01′	4°	50° 52° 54° 56° 58°	2° 34′ 2° 28′ 2° 21′ 2° 14′ 2° 07′	50° 04′ 52° 04′ 54° 04′ 56° 04′ 58° 04′

The student should plot this projection (Fig. 52) on 1 in. squared paper to the scale of 50′ (i.e. 50 geographical miles) to 1 in., taking 50° N. as origin on the central meridian, and marking off on this 52° as 120′ = 2·40 in., 54° = 4·80 in., 56° = 7·20 in., 58° = 9·60 in. from the origin. He will find that the result for this small region and small scale is indistinguishable from a simple conical projection. Actually the meridians are slightly curved, concave towards the central meridian.

Mercator's Projection.

In Vol. I it was explained how to calculate the distance of the parallels from the equator by enlarging the lengths of each short interval (2°) of latitude by multiplying it by the secant of the average latitude, and summing up such intervals for the total distance of the parallel from the equator. This is a close approximation to what mathematicians call "Integration," viz., finding $y = \int_0^\phi \sec \phi \, . R d\phi$ (where the symbol $\int$, the old English "long" S," means the sum of an infinite number of infinitely small increments). This "integral" is the value of $\sum_0^\phi \sec \phi \, . R\delta\phi$ where $\delta\phi$, the increment of latitude, is very small indeed, and can be shown by the calculus to be $R \log_e \tan \dfrac{90° + \phi}{2}$, so that the distance y of any parallel from the equator can be calculated directly and exactly. Tables of such distances are available,

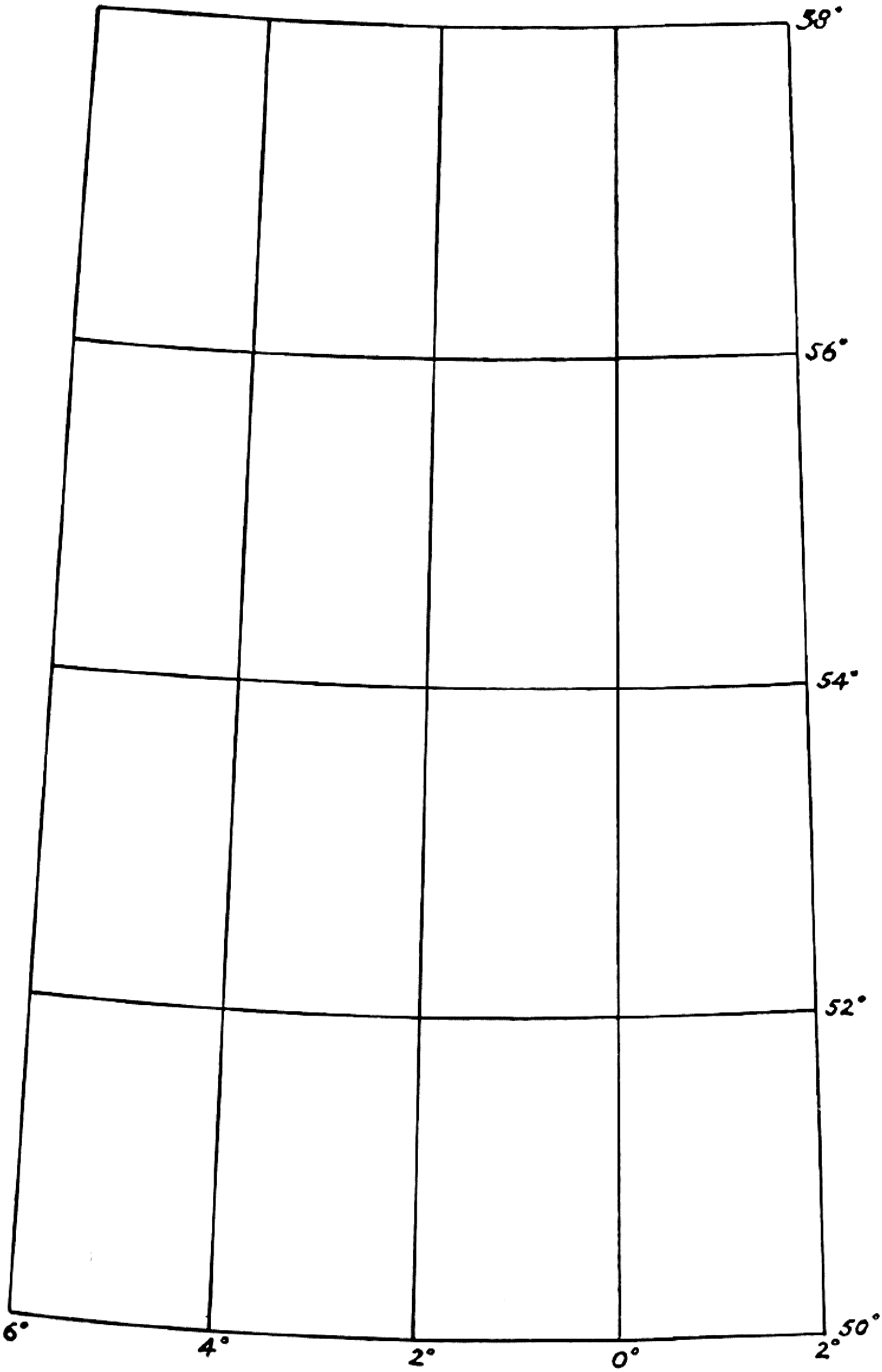


FIG. 52. CASSINI'S PROJECTION

e.g. in *Chambers's Seven-figure Mathematical Tables*, where they are called "Meridional Parts," and are expressed in minutes of angle, i.e. $y/R \times 3438$, there being 3,438 minutes in a radian, e.g. the meridional parts for every 10° of latitude are as follows—

Latitude	Meridional Part	Latitude	Meridional Part
0°	0·0 = 0° 0′	50°	3474·5 = 57° 54·5′
10°	603·1 = 10° 3·1′	60°	4527·4 = 75° 27·4′
20°	1225·1 = 20° 25·1′	70°	5966·0 = 99° 26·0′
30°	1888·4 = 31° 28·4′	80°	8375·3 = 139° 35·3′
40°	2622·7 = 43° 42·7	90°	Infinity

This means, for example, that the 70° parallel is plotted at a distance from the equator = 90° 26′ = 99·4333° on the scale used for longitude, whereas in Vol. I the figure arrived at was 9·94088 in. = 99° ·4088, a close agreement considering the comparatively large increment used.

The Transverse Mercator's or Gauss-Conformal Projection.

If in the above table of angular values of x and y for Cassini's projection we replaced each x by its "Meridional Part," keeping the y's unchanged we should convert the projection into an orthomorphic one, suitable like Cassini's for a region which does not extend far from its central meridian, and it has thus been used for Egypt. The scales parallel and perpendicular to the central meridian are always equal to each other, but are enlarged as the secant of the angular distance from the central meridian.

Transverse Zenithal Projections.

Transverse zenithal projections, on planes touching the sphere at the equator, are very easy to calculate, and the equal-area one makes a very good projection for large areas nearly bisected by the equator, e.g. Africa. The central meridian and the equator will be straight lines as great circles through the point of contact or centre of the map, and the other meridians and parallels will be symmetrical about these

lines. For each intersection A of parallels and meridians we must calculate the angular distance X and the azimuth α of A from the centre of the map O—(Fig. 53), where $AN = \phi$, the latitude, $ON = \theta$, the longitude, $\angle ANO = 90°$, $\angle AON = 90° - \alpha$, $OA = X$. Then $\cos X = \cos\theta \cos\phi$, and $\sin\theta$

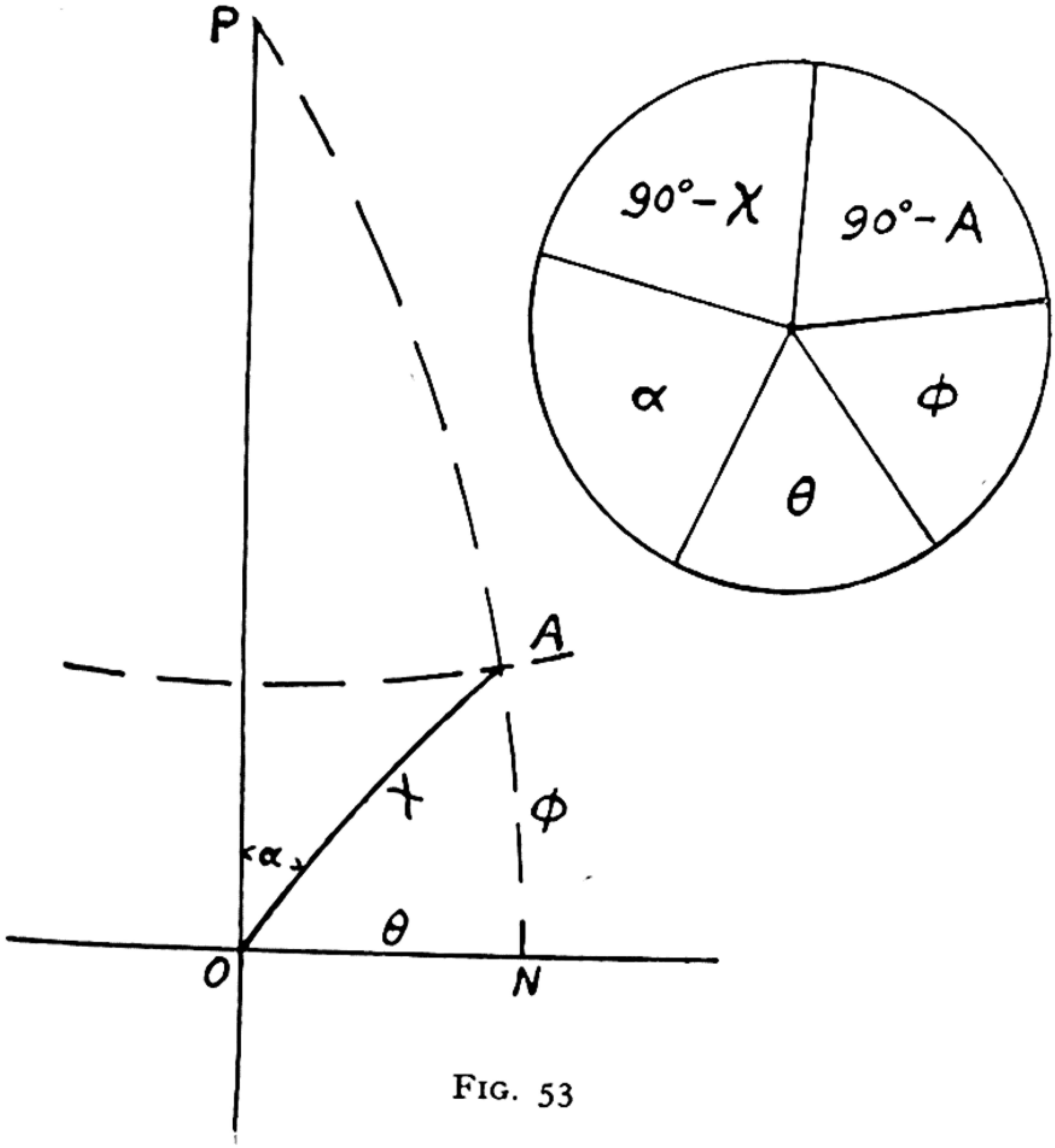


FIG. 53

$= \tan\alpha \tan\phi$, $\therefore \tan\alpha = \dfrac{\sin\theta}{\tan\phi}$. Then in our projection we draw from O a line making an angle α with the central meridian, and along it mark off from O a distance $r = RX$, $2R \sin\dfrac{X}{2}$, $2R \tan\dfrac{X}{2}$ or $R \tan X$ according as we require the Equidistant, the Equal-Area, Orthomorphic or Gnomonic projection. It is, however, more convenient and accurate to calculate the co-ordinates, viz., $y = r\cos\alpha$, $x = r\sin\alpha$, along

and perpendicular to the central meridian and to plot on squared paper. Each calculation will serve for four points, viz., those at the same longitude East and West of the central meridian, and at the same latitude North and South of the equator.

To calculate an equal-area zenithal projection for Africa, 40° S. to 40° N., 20° W. – 60° E., for a reduced earth of 5·73 in. radius on which 1 in. represents 10° of latitude (or longitude at the equator). The central meridian will be 20° E., and along it $\theta = O$, $\therefore X = \phi$, while along the equator $\phi = 0$, $\therefore X = \theta$, and the distances to be marked off along both these straight lines will be $5{\cdot}73 \times$ chd $10° = 1{\cdot}00''$, $5{\cdot}73 \times$ chd $20° = 1{\cdot}99''$, $5{\cdot}73$ chd $30° = 2{\cdot}97''$, $5{\cdot}73$ chd $40° = 3{\cdot}92''$. For the calculation of the co-ordinates of the other intersections we prepare a table with headings: θ, ϕ, log cos θ, log cos ϕ, log cos X, X, $r = R$ chd X; log sin θ, log tan ϕ, log tan a, a; x, y, and obtain the following results. Plotting the co-ordinates we draw the curved parallels and meridians as in Fig. 54—

θ	ϕ	X	r	a	x	y
			Inch		Inch	Inch
10°	10°	14° 03′	1·40	44° 34′	0·98	1·00
	20°	22° 15′	2·21	25° 30′	0·95	2·00
	30°	31° 28′	3·10	16° 45′	0·89	2·97
	40°	41° 01′	4·02	11° 42′	0·82	3·93
20°	10°	22° 15′	2·21	62° 44′	1·96	1·01
	20°	27° 58′	2·76	43° 13′	1·89	2·01
	30°	35° 32′	3·50	30° 39′	1·78	3·01
	40°	43° 57′	4·29	22° 11′	1·62	3·97
30°	10°	31° 28′	3·10	70° 34′	2·92	1·03
	20°	35° 32′	3·50	53° 57′	2·83	2·06
	30°	41° 25′	4·05	40° 54′	2·65	3·06
	40°	48° 26′	4·70	30° 48′	2·40	4·04
40°	10°	41° 01′	4·02	74° 40′	3·88	1·06
	20°	43° 57′	4·29	60° 29′	3·73	2·11
	30°	48° 26′	4·70	48° 04′	3·50	3·14
	40°	54° 03′	5·21	37° 27′	3·16	4·14

As regards the properties of this projection, the direction a of every point is correctly shown from the centre of the map, but the scale in the *radial* direction from the centre is

reduced in the ratio $\cos\frac{X}{2} : 1$, and increased in the *circumferential* direction at right angles thereto as $\sec\frac{X}{2} : 1$, as shown in Vol. I. At the extreme points, $\theta = 40°$, $\phi = 40°$, we have

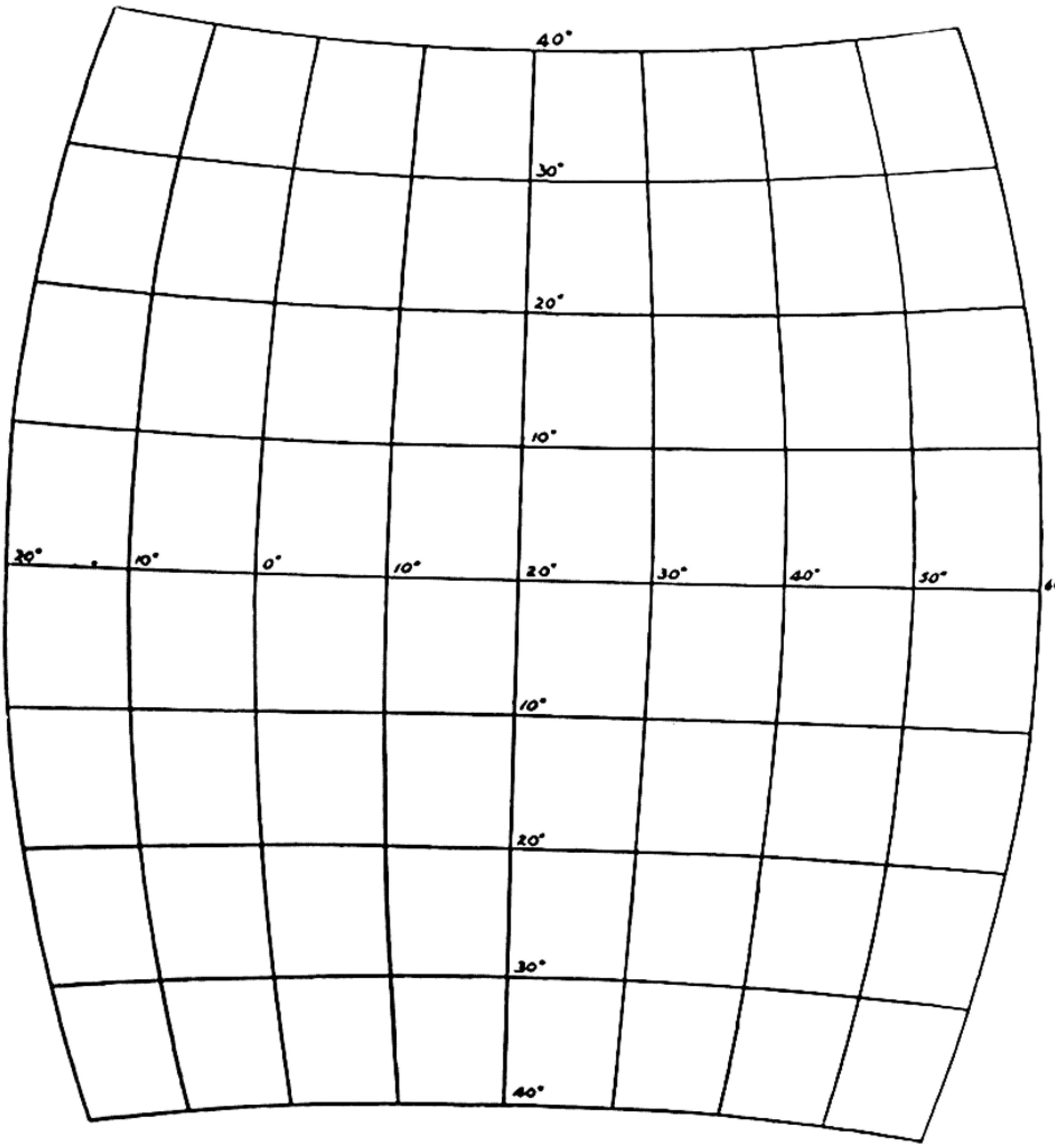


FIG. 54. EQUAL-AREA ZENITHAL

$X = 54° \ 03'$; these ratios become ·891 and 1·122. Very little of the continent itself, however, extends beyond $\theta = 30°$, $\phi = 30°$, at which points $X = 41° \ 25'$, and the scale ratios become ·935 and 1·069, so that the scale errors here do not exceed 7 per cent, while at the centre of the map the scale is correct in all directions. The projection is, therefore, a very

suitable one for Africa, and is now much used for the world in hemispheres, though round the edge of the hemisphere

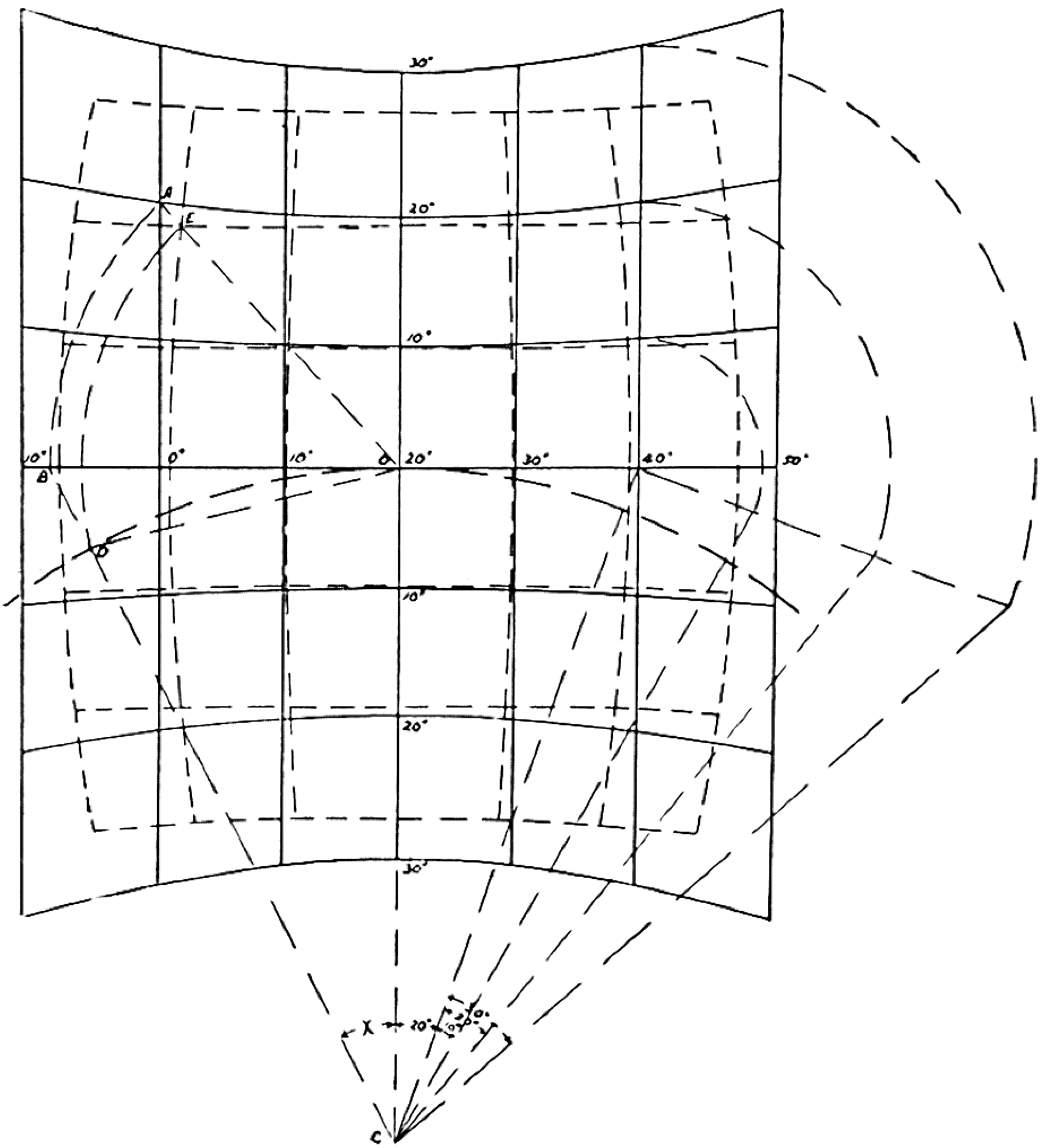


FIG. 55

$X = 90°$, and the scale ratios are, therefore, radial, cos $45° = \cdot 707$; circumferential, sec. $45° = 1{\cdot}414$.

The transverse stereographic is readily constructed graphically as shown in Vol. I, as is also the transverse gnomonic, also given in Vol. I, or the latter can be drawn from the calculated co-ordinates of the intersections as, obviously, $x = R \tan \theta$,

$y = R \sec \theta \tan \phi$ from the construction. The transverse gnomonic can be easily transformed into the equal-area zenithal as the azimuths of the intersections from the centre are the same in both projections (being true azimuths), and the linear distances have only to be reduced in the ratio chd X to tan X, as shown in Fig. 55. We describe a circle, centre O, radius OA, to cut the equator at B, join B to C, the centre of the circle of radius R, cutting this circle at D. Then $\angle DCO = X$, $OD = R$ chd X, and we make $OE = OD$ on line OA, and repeat this for each intersection, then draw the parallels and meridians through the points thus found.

Oblique Zenithal Projections.

When the "centre of the map" is not at the pole or on the equator, e.g. for a map of Asia, the central meridian alone will be a straight line, and the other meridians and parallels will be symmetrical about it, so that the calculations will in each case only serve for two points at the same longitude and latitude on each side of the central meridian. The spherical triangles to be solved are now *oblique*, viz., BCA (Fig. 56) where B is the centre of the map at latitude ϕ_o, C the pole and A the intersection, in which triangle we have $a = BC = 90° - \phi_0$, $b = CA = 90° - \phi$, and angle $C = \theta$, the longitude-difference from the central meridian, while we require to find $BA = c = X$, the angular distance from B, and $\alpha = B$, the azimuth of A from B. We could find these in a similar way to Example 1 in the last chapter by dropping a perpendicular BN to CA, but it is quicker to use the formulae for oblique triangles, viz., $\cos X = \sin \phi_o \sin \phi + \cos \phi_o \cos \phi \cos \theta$, and then to find α by $\sin \alpha = \dfrac{\cos \phi \sin \theta}{\sin X}$. (See Note on page 153.)

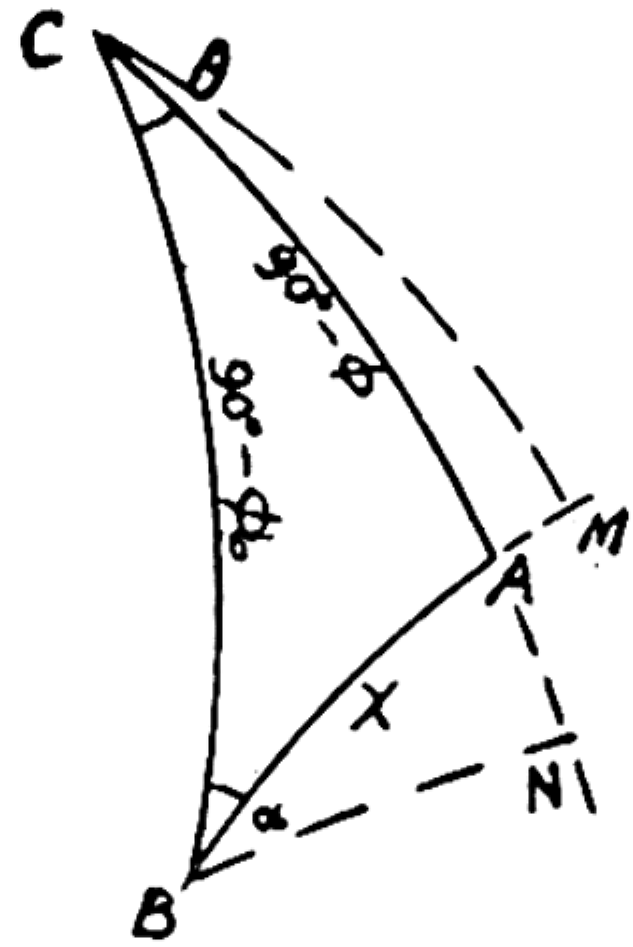


FIG. 56

Cartographical tables are available, such as those given briefly in *Hinks' Map Projections*, which give the angular distances X and the azimuths α of the intersections of parallels and meridians from the centre of the map at various latitudes.

The angular distance X requires merely to be converted into $r = RX$, R chd X, $2R \tan \frac{X}{2}$ or $R \tan X$, and the co-ordinates

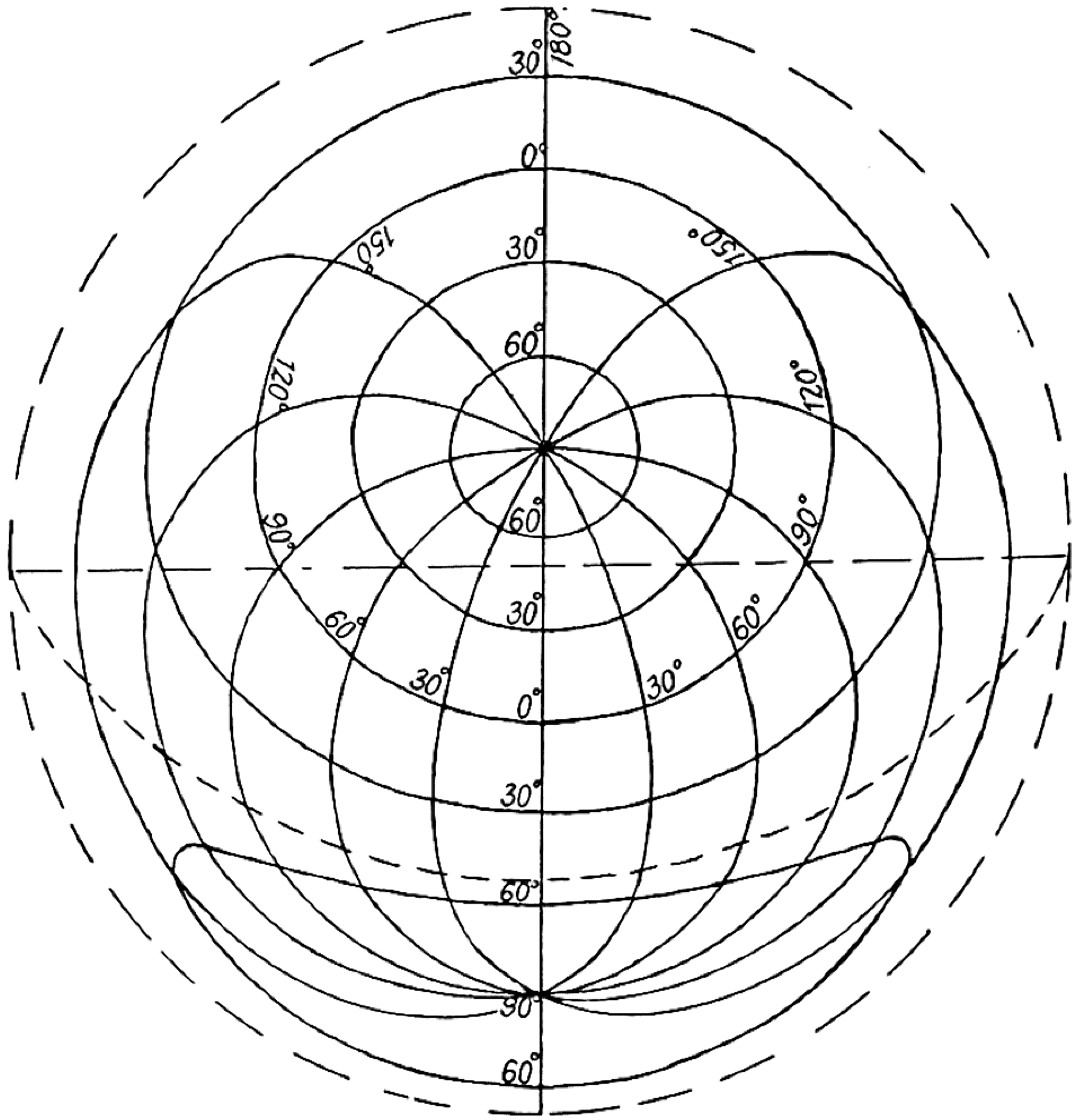


FIG. 57

y and x along and perpendicular to the central meridian are then $r \cos \alpha$ and $r \sin \alpha$.

The zenithal equal-area projection is probably the best for the continents for general purposes. The zenithal equidistant is useful for wireless telegraphy and aeronautics, as it gives directly the direction and shortest distance of all points on the earth's surface from the place chosen as the centre of the map. Fig. 57 shows a zenithal equidistant of the whole world with London (latitude 51° 30′ N.) as centre. The enclosing

circle represents the antipodes of London (51° 30′ S., 180° E.), which is, of course, equally distant from London in all directions.

The zenithal orthomorphic (stereographic) can be drawn with its centre at any latitude by purely geometrical methods,

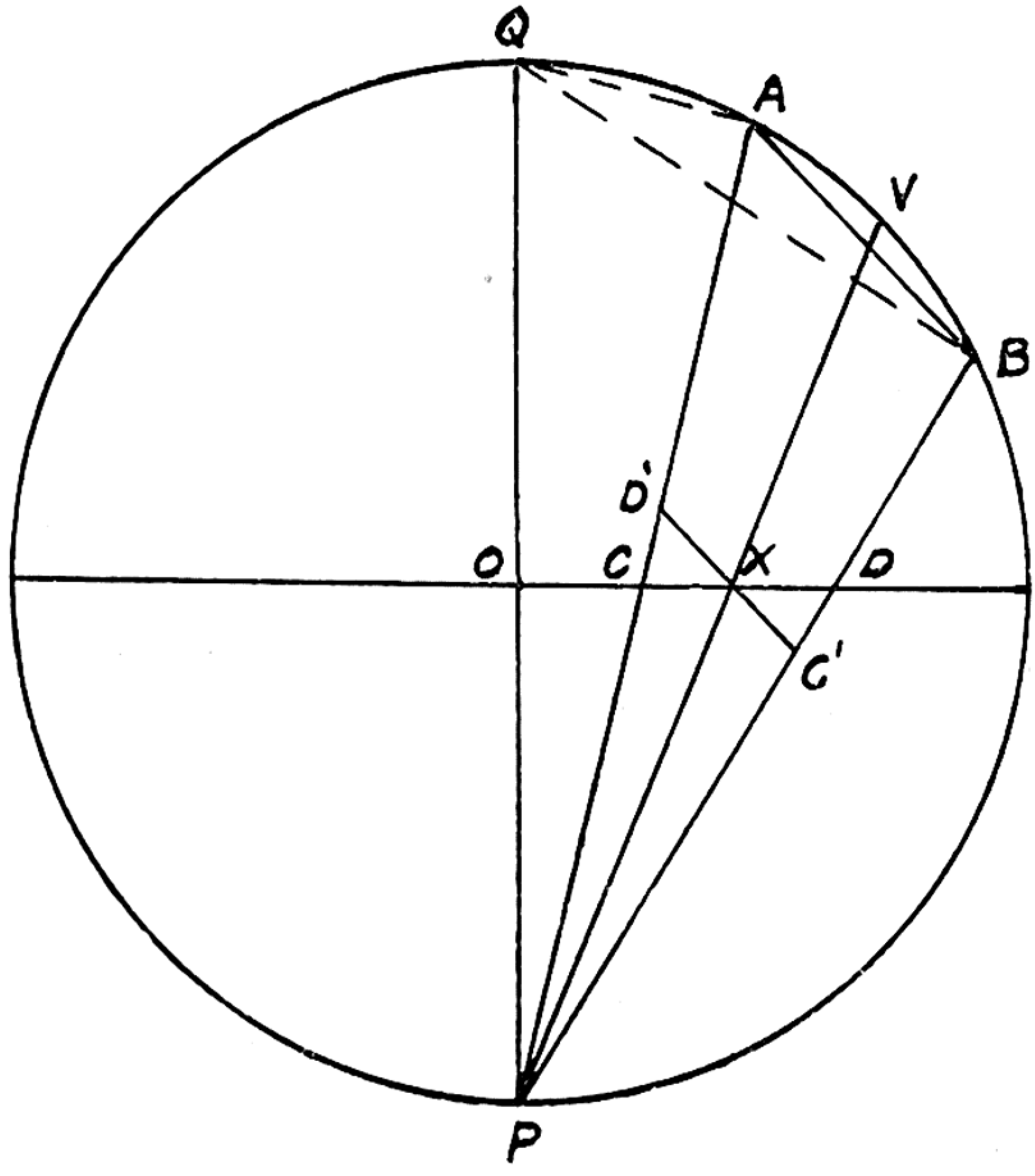


FIG. 58

owing to its peculiar property that the projection of any circle (great or small) on the earth is a circle on the map. To prove this: Let AB (Fig. 58) represent the elevation of a small circle in the sphere, V being its "pole," the circle $QAVBP$ being the great circle through V and P, and P the point of projection. CD represents the projection on the plane through O, the centre of the sphere, parallel to the tangent plane at Q, and we shall consider the projection made on this plane. The projection will, obviously, be just half the scale that it

would have been in the tangent plane. The rays from P to the small circle AB form an oblique cone, whose section by the plane AB, perpendicular to the plane of the paper, is a circle. We can assume that the section of this cone by the plane CD is an ellipse, as it is that of a circle viewed obliquely, and from symmetry we can see that this ellipse will be symmetrical about CD. The line VP bisects the angle APB, as the arcs VA, VB are equal. Also, $AP \,.\, CP = PQ \cos QPA\ OP \sec QPA = PQ \,.\, OP$; similarly, $BP \,.\, DP = PQ \,.\, OP = AP \,.\, CP$. $\therefore \dfrac{CP}{BP} = \dfrac{DP}{AP}$. If we rotate the cone CDP round the line VP so that it lies in the position $C'D'P$, we have $\dfrac{C'P}{BP} = \dfrac{D'P}{AP}$, and, therefore, $C'D'$ is parallel to AB, and $C'D' = CD = AB\dfrac{C'P}{BP} = AB\dfrac{CP}{BP}$. Let CD and $C'D'$ cut at X; by symmetry X lies on the line VP and $XC = XC'$. The breadths at X of the sections of the cone through CD and $C'D'$, perpendicular to the plane of the paper, are, therefore, the same, and, therefore, these two sections are identical in shape and size. That through $C'D'$ being parallel to the small circle AB is a circle. Therefore, the section CD is also a circle of diameter $= AB\dfrac{CP}{BP}$.

To Construct a Stereographic Projection for Asia, 20° *E. to* 180° *E.,* 70° *N.* – 10° *S., with its Centre at* 100° *E.,* 30° *N.* (Fig. 59). Draw a circle of radius $2R$, twice that of the reduced earth, as we shall take our projection, as in the previous paragraph, on a plane through the centre of the sphere. Divide the circumference into 10° intervals. Draw the diameter from 30° N. to 30° S. and a diameter QQ' at right angles to it, which will represent the plane of projection and the central meridian of 100° E. Joining the point 30° S. to 90° N. and 90° S., we find P, P' the north and south poles on QQ'. For any parallel, 70° N., say, we join 30° S. to the two points 70° N., cutting QQ' at A and A' and on AA' as diameter draw a circle which gives the parallel of 70° N., and so for the other parallels. The meridians will all be circular arcs through P and P', their centres will, therefore, all be on the line BB' bisecting PP' at right angles. As the projection is orthomorphic the

meridians will meet the central meridian at true angles at P and P'. For example, the meridians of 20° E. and 180° E. will have their tangents at P inclined at angles of 80° to the central meridian PP'. Their centres C, C' can thus be found

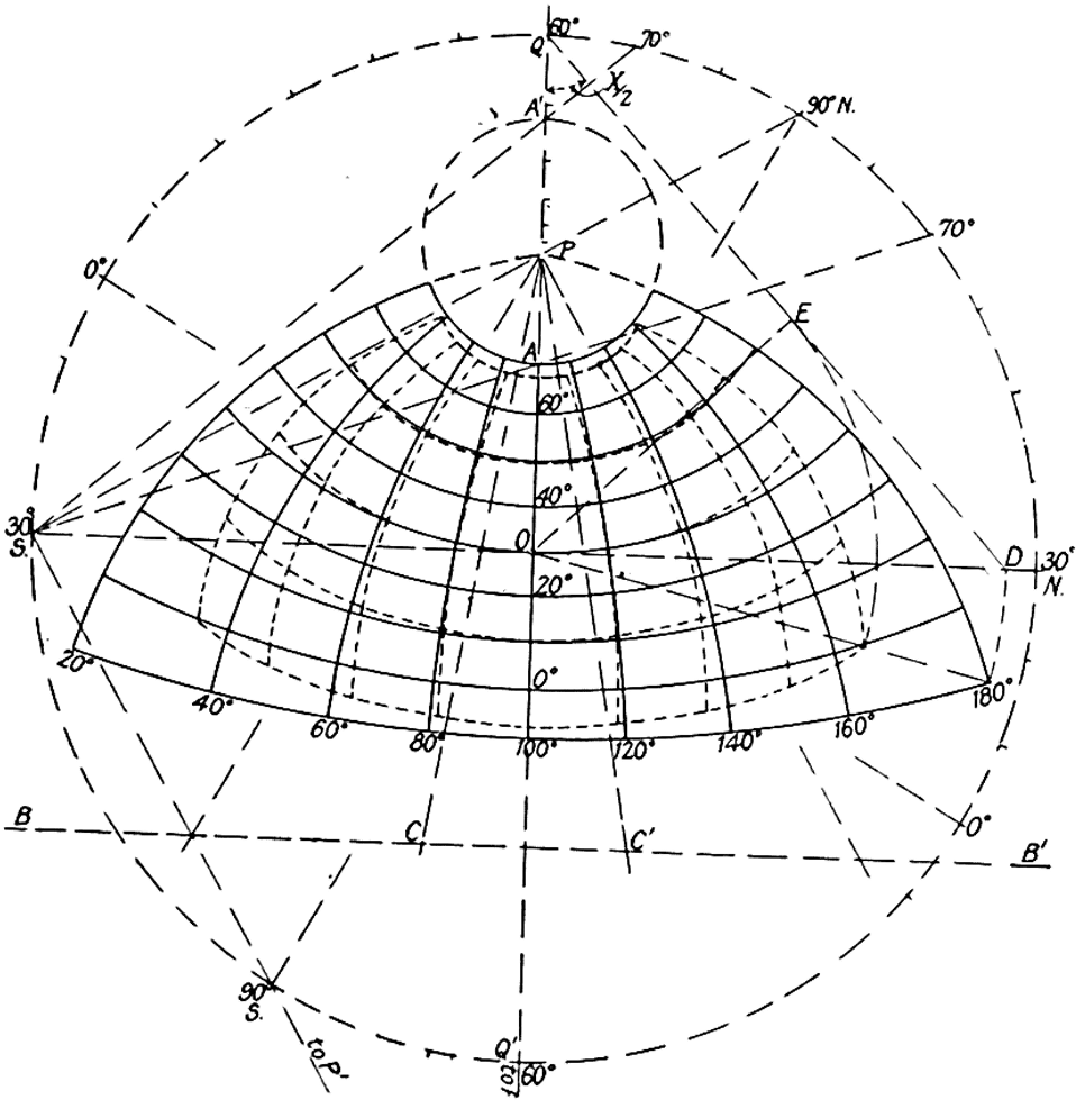


FIG. 59

by drawing radii inclined at 10° to PP', meeting BB' at C and C', and similarly for the other meridians.

The meridians and parallels will be found to cut everywhere at 90°, but the great distortion round the edges of the projection will be readily seen, the intervals between the parallels and between the meridians becoming rapidly greater as we travel from the centre of the map. The stereographic projection is in itself of purely theoretical interest—the fact that

the meridians and parallels are all circular arcs is of no use in maps of any considerable scale as the radii becomes very large. It is of no use for navigation, as the meridians are not straight and parallel, and the large exaggeration of scale neutralizes any advantages afforded by its orthomorphism.

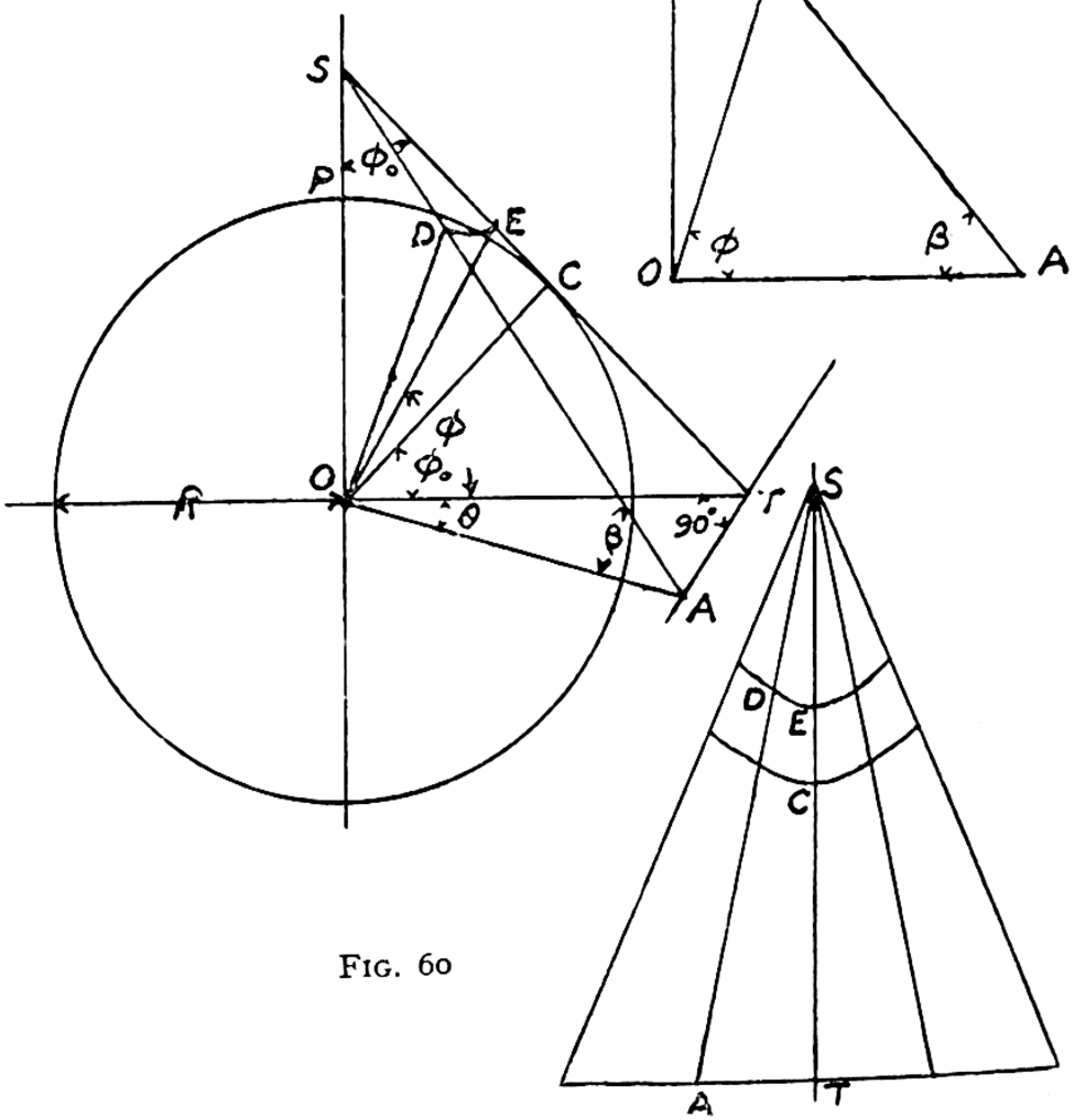


FIG. 60

Thus at an angular distance of 41° 25′ from its centre the scale exaggeration is $\sec^2 \frac{41^\circ\ 25'}{2} = (1{\cdot}0690)^2 = 1{\cdot}143$, or more than double that for the zenithal equal-area, while the exaggeration of areas here will be $(1{\cdot}143)^2 = 1{\cdot}307$, or over 30 per cent. It is, however, instructive to convert it into the

zenithal equal-area by graphical construction, as shown on Fig. 59, for the point 180° E., 10° S. The azimuth of this point from the centre O will be unaltered, as it is the true azimuth, but its distance from O is to be reduced in the ratio $2R \sin \frac{X}{2} : 2R \tan \frac{X}{2}$. Lay off the stereographic distance OD from the centre O at right angles to OQ and join DQ. This gives us the angle $DQO = \frac{X}{2}$. Then the perpendicular distance OE from O to DQ gives us $2R \sin \frac{X}{2}$ = the distance to mark off along the azimuth line for the point on the equal-area projection. The greatly improved shape of the net will be readily seen, as shown by the dotted lines.

The Oblique Gnomonic Projection.

The oblique gnomonic projection can also be constructed without spherical trigonometry, with its centre at any latitude ϕ_o (Fig. 60). From the centre C set off $CS = R \cot \phi_o$, and in the opposite direction $CT = R \tan \phi_o$, both along a straight central meridian. Draw $TA \perp CT$ to represent the equator. Then $OT = R \sec \phi_o$, $AT = R \sec \phi_o \tan \theta$, mark AT along the equator and join AS to represent the meridian θ from the central meridian. S represents the pole. Also, $OA = R \sec \phi_o \sec \theta$, and $\angle SAO = \beta$ is given by $\tan \beta = \frac{SO}{OA} = \frac{R \operatorname{cosec} \phi_o}{R \sec \phi_o \sec \theta} = \cot \phi_o \cos \theta$. Then in the triangle DOA we have $AD = \frac{OA \sin \phi}{\sin (\phi + \beta)} = \frac{R \sec \phi_o \sec \theta \sin \phi}{\sin (\phi + \beta)}$, which we mark off along meridian θ from the equator for the parallel ϕ. A gnomonic projection for the North Atlantic Ocean with its centre 45° N., 40° W. is shown in Fig. 61. The geometrical construction, which may be employed instead of trigonometry, is obvious (see Fig. 73).

The Cubic Gnomonic Projection.

The peculiar property of the gnomonic projection is that all great circles on the sphere are represented by straight lines on the map and vice versa. It is, therefore, useful for finding the shortest courses for air or water navigation between two points, but does not give the distances nor directions. As

it is impossible to represent a hemisphere on the projection while the distortion becomes very great when the angular distance, X, from the centre exceeds 45°, the projection is effected on the six faces of a cube touching the sphere, two at the poles and the remaining four at the equator (Fig. 62).

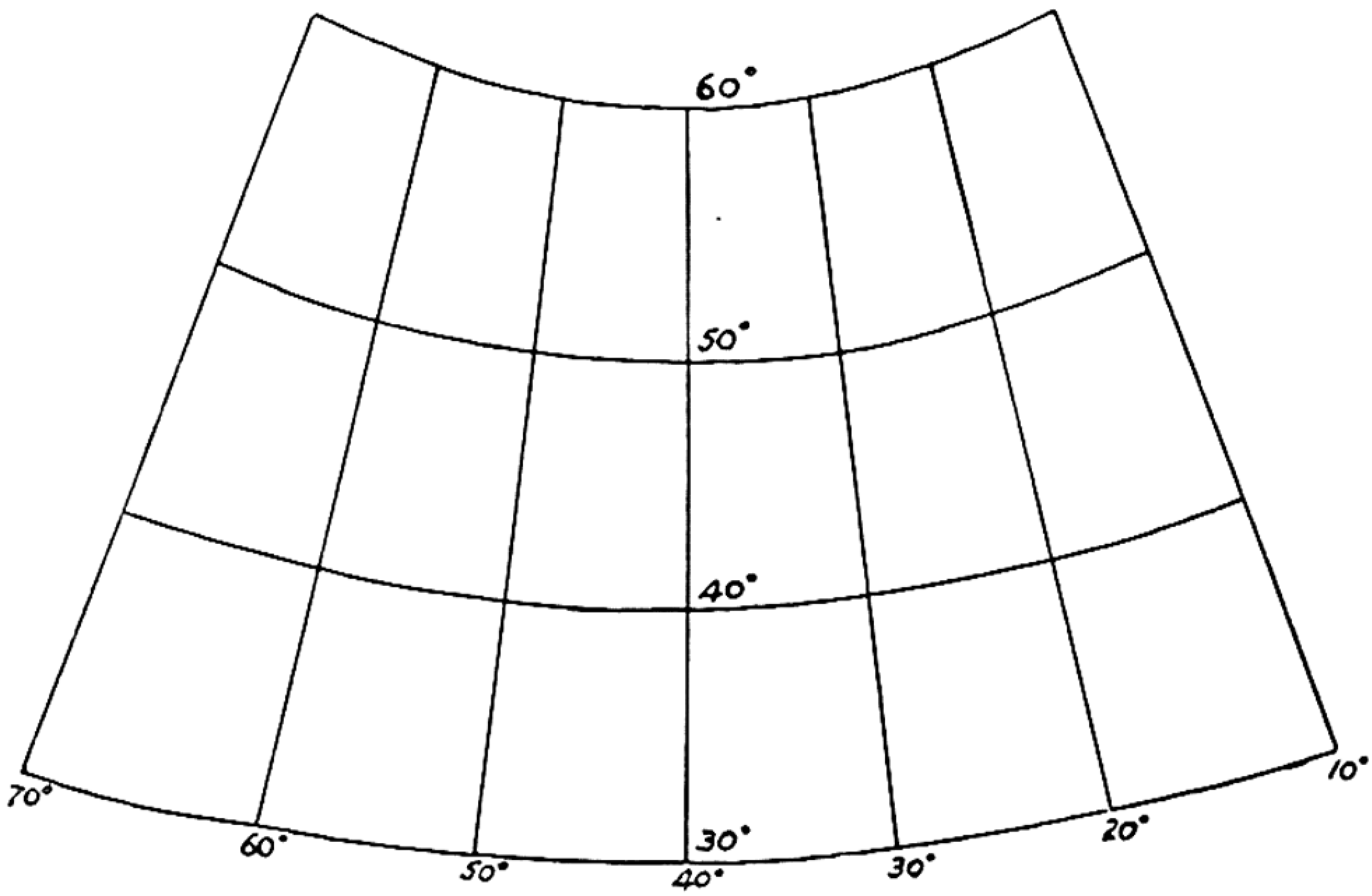


FIG. 61

For the polar faces we have $r = R \tan X$, where X is the co-latitude, for the equatorial faces $x = R \tan \theta, y = R \sec \theta \tan \phi$, where θ, ϕ are difference of longitude and latitude. Taking $R = 1$ in., we find the polar radii and values of x as (10°) ·176 in., (20°) ·364 in., (30°) ·577 in., (40°) ·839 in., (50°) 1·192 in., and for the y's we prepare a table with headings as follows—

$$\theta,\ \phi,\ \log \sec \theta,\ \log \tan \phi,\ \log y,\ y$$

and obtain the following results, plotted in Fig. 62—

θ	ϕ	y	θ	ϕ	y	θ	ϕ	y	θ	ϕ	y	θ	ϕ	y
		Inch			Inch			Inch			Inch			Inch
10°	10°	·179	20°	20°	·188	30°	10°	·204	40°	10°	·230	45°	10°	·249
	20°	·370		20°	·387		20°	420		20°	475		20°	·515
	30°	·586		30°	·614		30°	·667		30°	·754		30°	·816
	40°	·852		40°	·893		40°	·969		40°	1·095		40°	1·187

For drawing great circle courses there are three cases: (*a*) When both the points are on the same face of the cube we merely join them by a straight line. (*b*) When the two points are on opposite faces we make use of the fact that all great

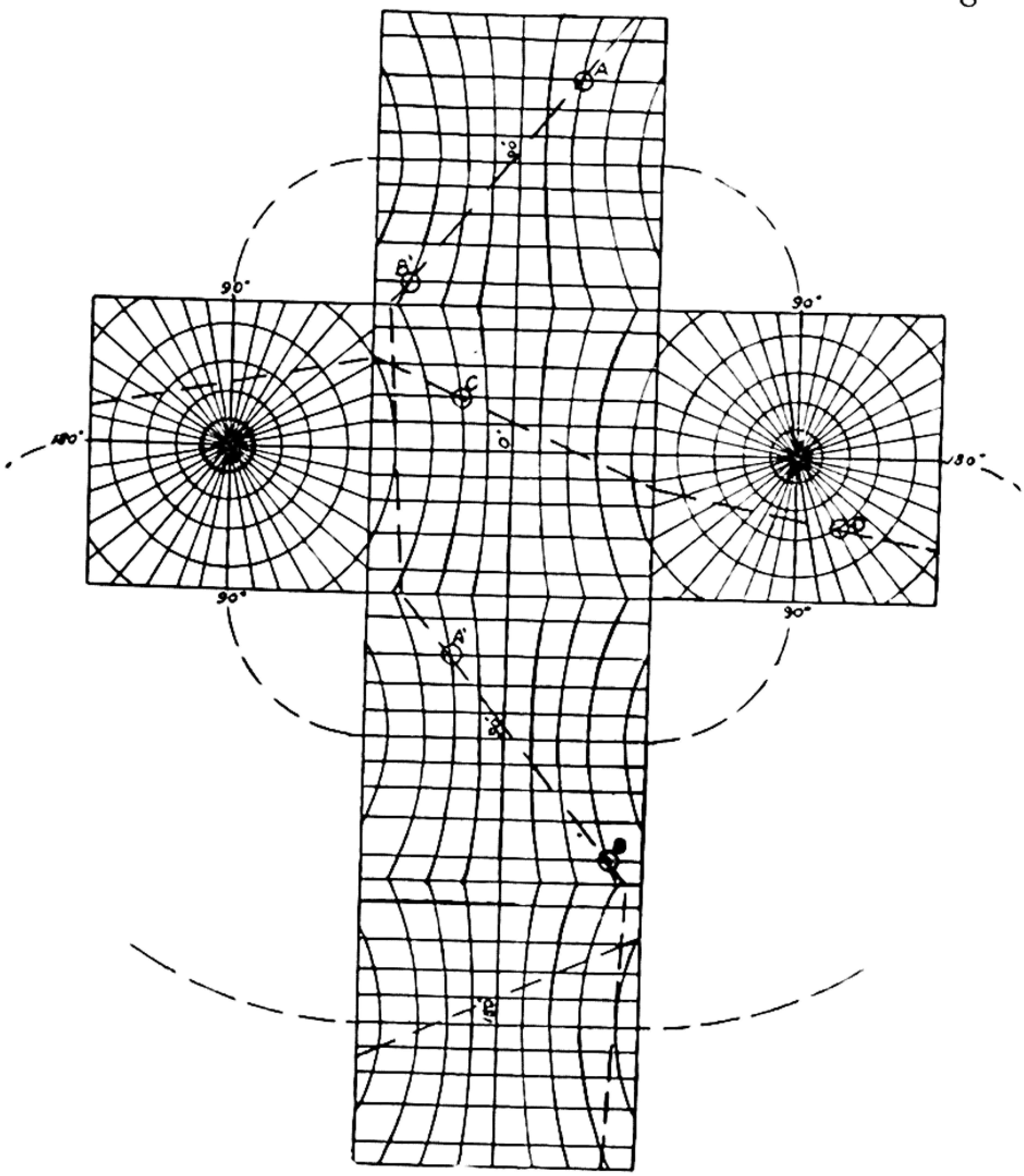


FIG. 62

circles through a point pass through the antipodes of the point. Referring to Fig. 62, *A* is at 120° E., 20° S., *B* at 130° W., 30° S.; and we mark *A*′ at 60° W., 20° N., and *B*′ at 50° E., 30° N. Then we join *A* and *B*′ also *A*′ and *B*, and continue the line round the intervening faces as shown. (*c*) When the two points *C* and *D* in Fig. 62 lie on adjacent faces we use the

fact that the outer edges of the pair of adjacent faces represent opposite parts of the same great circle, and the great circle through C and D must meet these edges at diametrically opposite points, i.e. points which are at equal distances from the common centre line of the faces, but measured in opposite directions. We choose a point on the common edge, and draw lines through C and D to the outer edges—if the points just

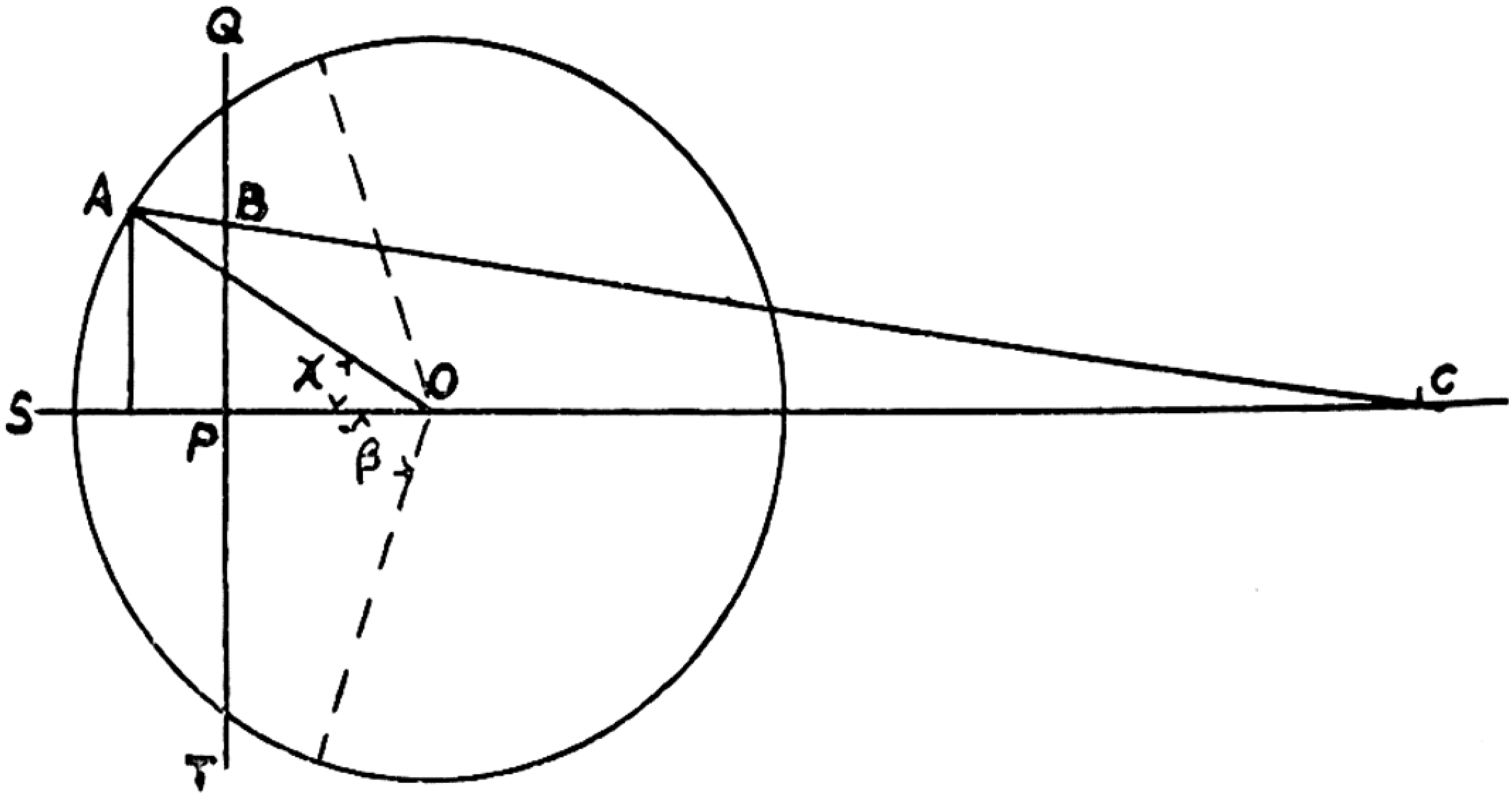


FIG. 63

found do not conform to the above condition we adjust the point on the common edge until they do so, as in Fig. 62.

Perspective Projections.

The gnomonic and the stereographic are cases of "perspective" projections, points on the sphere being projected on to a plane by rays all passing through one point, as if the sphere were transparent and the eye placed at the "centre of projection." If the centre of projection is taken outside the sphere (Fig. 63) at a distance, $OC = hR$ from the centre of the sphere, and the projection is made on a plane QPT perpendicular to OC, between O and S where CO produced cuts the sphere, so that $CP = kR$, where R is the radius of the reduced earth, we have the following relation, X being the angular distance of a point A on the sphere from S, and $r = BP$, the linear distance of its projection

B from P, the centre of the map, $\dfrac{r}{R \sin X} = \dfrac{kR}{hR + R\cos X}$ $= \dfrac{k}{h + \cos X}$. $\therefore r = \dfrac{kR \sin X}{h + \cos X}$. This gives us the radial distance, r, of the projection B from the centre of the map P, and, as in the zenithal projections, the azimuth of B will be the true azimuth on the sphere, as great circles through C and S cut the plane QPT at true angles. The circumferential scale, t, is of course, $\dfrac{r}{R \sin X} = \dfrac{k}{h + \cos X}$ (which becomes $\dfrac{k}{h + 1}$ at the centre of the map). The radial scale, s, can be found by "differentiating" to be $\dfrac{dr}{RdX} = \dfrac{k(h \cos X + 1)}{(h + \cos X)^2}$ (which becomes $\dfrac{k(h + 1)}{(h + 1)^2} = \dfrac{k}{h + 1}$ at the centre of the map).

Clarke's Projection.

In Clarke's projection the numbers h and k are so chosen that the "total misrepresentation" is a minimum, i.e. that the sum of the squares of the scale errors in the radial and circumferential directions is a *minimum*, taken for every point on the area projected. Expressed in the notation of the calculus this condition is that $\int_0^\beta \{(s-1)^2 + (t-1)^2\} \sin X dX$ is a minimum, where β is the angular radius of the area to be mapped as s and t are the same for a ring of area $2\pi R^2 \sin XdX$ of the sphere. This makes a good projection for Asia, a very difficult continent for the cartographer, as it extends 25° E. to 170° W., 78° N. to 11° S., for a hemisphere and even for larger areas. It is not, of course, either equal area or orthomorphic, but a good average between the two. For a hemisphere, $k = 2{\cdot}034$, $h = 1{\cdot}47$, $\beta = 90°$, and the scale in both directions at the centre is $\dfrac{2{\cdot}034}{2{\cdot}47} = {\cdot}823$. At the edge the radial scale is $\dfrac{2{\cdot}034}{1{\cdot}47^2} = {\cdot}941$, and the circumferential scale $= \dfrac{2{\cdot}034}{1{\cdot}47} = 1{\cdot}384$. In the zenithal equal-area both scales are

1·000 at the centre and ·707 and 1·414 at the circumference of the hemisphere, so that the extreme errors are reduced at the expense of the accuracy near the centre of the map.

The Orthographic Projection.

The Orthographic projection is a perspective projection with the centre of projection at an infinite distance, so that the

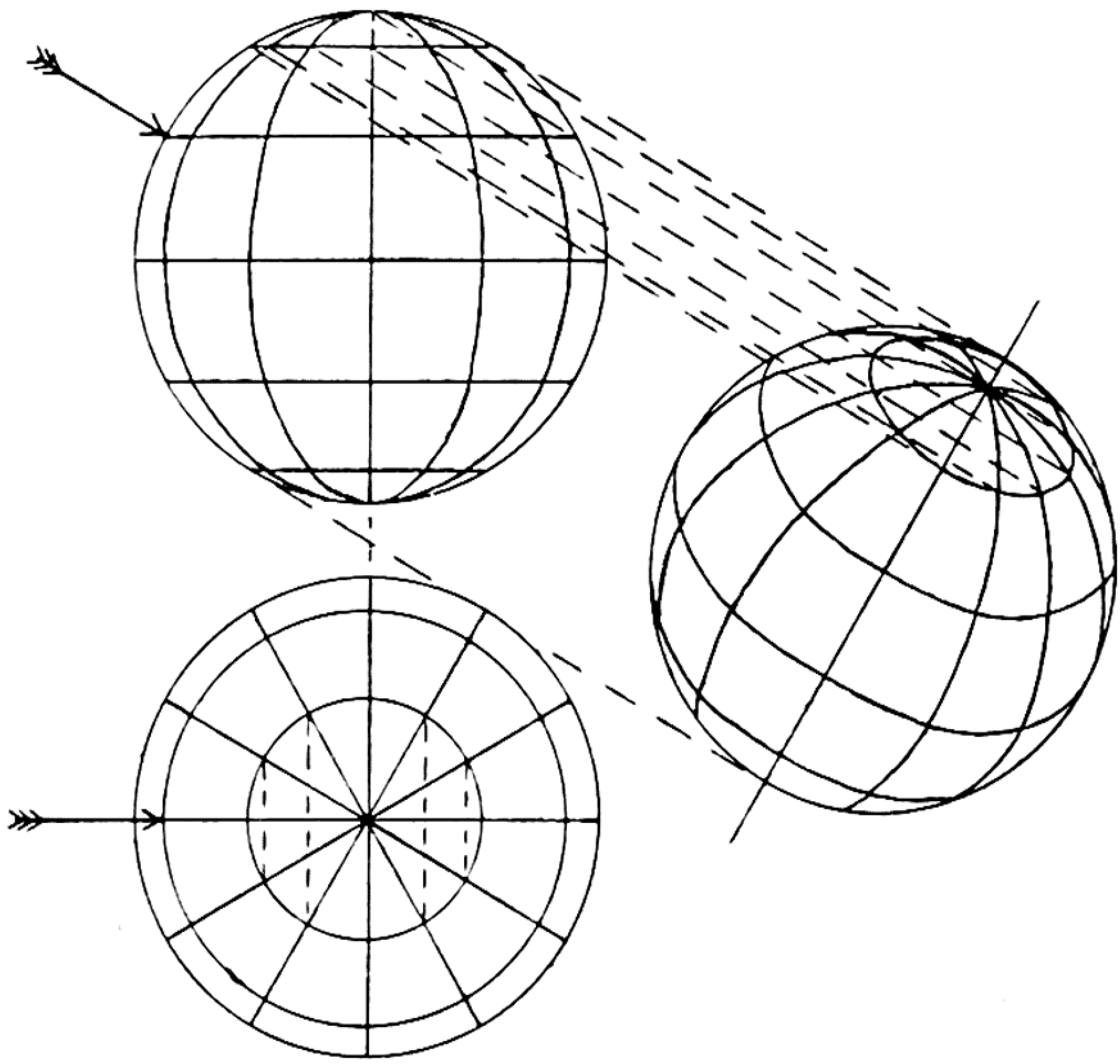


FIG. 64

eye is supposed to be opposite every point of the hemisphere at the same time, as in an engineer's or architect's elevation. In practice it is only of use for astronomical purposes, such as transits of Venus. The polar and transverse cases of this projection have been illustrated in Vol. I. To draw an *oblique projection* with its centre at 30° N. we proceed as follows (Fig. 64): Having drawn the polar and transverse projections, we project all the intersections of parallels and meridians from the transverse projection on to a line at 90° to the direction of projection—then on each side of this line set out the

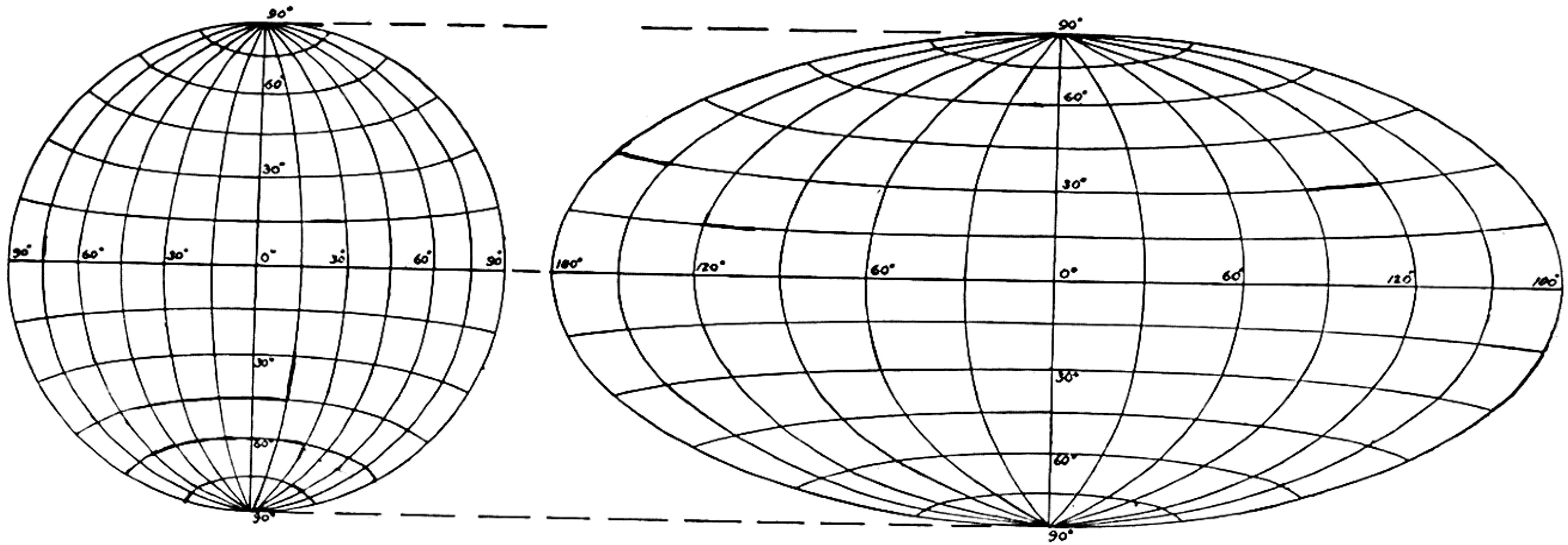

90°
60°
30°
0°
30°
60°
90°
60°
30°
0°
30°
60°
90°
90°
60°
30°
0°
30°
60°
90°
180°
120°
60°
0°
60°
120°
180°

FIG. 65

distance (on the polar projection) of the point from the meridian of the centre of the map, and join the points thus found to form the parallels and meridians, all of which are, of course, ellipses.

Aitoff's Projection (Fig. 65).

The zenithal equal-area projection is drawn (or calculated) for a hemisphere with the centre of the map on the equator. The projection is then pulled out uniformly parallel to the equator, so that its outline is an ellipse of major axis twice the minor axis, i.e. the distances of all points from the central meridian are doubled, while their distances from the equator are left unchanged. Then the value of each meridian is doubled, e.g. 90° becomes 180°.

The projection is obviously equal-area, and the parallels are more perpendicular to the meridians than in Mollweide's projection ; the effect is, therefore, more pleasing.

CHAPTER VIII

MORE ADVANCED CONICAL PROJECTIONS

Simple Conical Projection with Two Standard Parallels.—Choice of Standard Parallels.

In Vol. I, page 106, we calculated this projection for the region 40° – 70° N., 20° to 60° E., *arbitrarily* choosing our standard parallels as 45° and 65°, or at one-sixth the range of latitude from the extreme parallels, and we found the

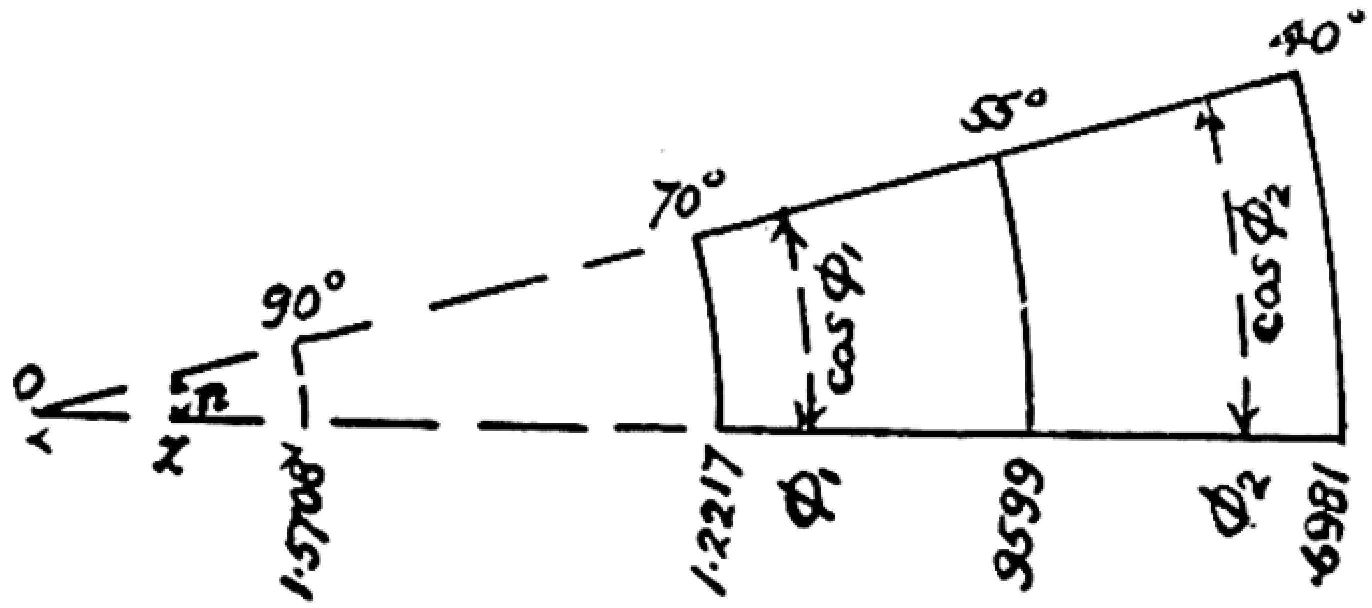


Fig. 66

percentage scale errors along the parallels of 40°, 55°, and 70° as + 1·6 per cent, – 1·7 per cent, and + 2·9 per cent respectively. We can improve the accuracy of the map substantially if we select the two standard parallels, so that the errors along the extreme and central parallels shall all be equal. To do this we proceed as follows: Let n be the "constant" of the cone, i.e. the ratio that the angles between the meridians on the map make to their true angles on the sphere, then, if ϕ_1, ϕ_2 are the standard parallels and r_1, r_2 their radii on the map and θ be any longitude difference, $n\,(r_2 - r_1)\,\theta = R(\cos\phi_2 - \cos\phi_1)\theta$.

$$\therefore n = \frac{R(\cos\phi_2 - \cos\phi_1)}{r_2 - r_1} = \frac{\cos\phi_2 - \cos\phi_1}{\phi_1 - \phi_2}$$

in our case, as $r_2 - r_1 = R(\phi_1 - \phi_2)$. To determine the two standard parallels we can simplify matters by assuming in Fig. 66 that $\theta = 1$, and $R = 1$; let us call the radius of the

polar arc z. Then as we have to make the scale errors at 70°, 40°, and 55° all equal, we have

$$\frac{n(z + \pi/2 - \text{arc } 70°)}{\cos 70°} - 1 = \frac{n(z + \pi/2 - \text{arc } 40°)}{\cos 40°} - 1$$
$$= 1 - \frac{n(z + \pi/2 - \text{arc } 55°)}{\cos 55°}$$

i.e. $$\frac{n(z + 1{\cdot}5708 - 1{\cdot}2217)}{{\cdot}3420} - 1 = \frac{n(z + 1{\cdot}5708 - {\cdot}6981)}{{\cdot}7660} - 1$$
$$= 1 - \frac{n(z + 1{\cdot}5708 - {\cdot}9599)}{{\cdot}5736}$$

$$\therefore \frac{nz + 0{\cdot}3491n}{{\cdot}3420} - 1 = \frac{nz + 0{\cdot}8727n}{{\cdot}7660} - 1 = 1 - \frac{nz + 0{\cdot}6109n}{{\cdot}5736}$$

$\therefore$ ${\cdot}4240\,nz = 0{\cdot}03105n$. $\therefore$ $z = 0{\cdot}0732$ from the first equation.

Substituting in the second equation we have $n = {\cdot}8239$.

For the standard parallels we must have $n\left(z + \frac{\pi}{2} - \phi_1\right) = \cos \phi_1$; $n\left(z + \frac{\pi}{2} - \phi_2\right) = \cos \phi_2$. $\therefore$ ${\cdot}8239\ (1{\cdot}6440 - \phi_1) = \cos \phi_1$ and ${\cdot}8239\ (1{\cdot}6440 - \phi_2) = \cos \phi_2$, and the values of ϕ_1, ϕ_2 to satisfy these must be found by trial. They are found to be 66° 48′ and 45° 09′ respectively. To find the percentage error on the extreme and central parallels, we must substitute the above values of n and z in any of the original expressions, and we find the scale ratio is 1·0174. Therefore, the errors are all 1·74 per cent. The radii of the standard parallels will be $r_1 = R\ ({\cdot}0732 + 1{\cdot}5708 - 1{\cdot}1653) = {\cdot}4787R$, and $r_2 = ({\cdot}0732 + 1{\cdot}5708 - {\cdot}7880) = {\cdot}8560R$. There will be a rather larger error on a parallel near to 55°, and it is possible so to choose the standard parallels that this maximum error will be the same as on the extreme parallels, but the process is complicated, and it is not generally worth the trouble.

The Rectangular Polyconic.

The ordinary polyconic, which is described in Vol. I, and which has been modified into the international map, may itself be described as a modified conical projection in which

every parallel is a standard parallel in radius as well as in length. The scale along every parallel is uniform and true, but the scale along the meridians is only true along the central meridian, as the parallels are not concentric circular

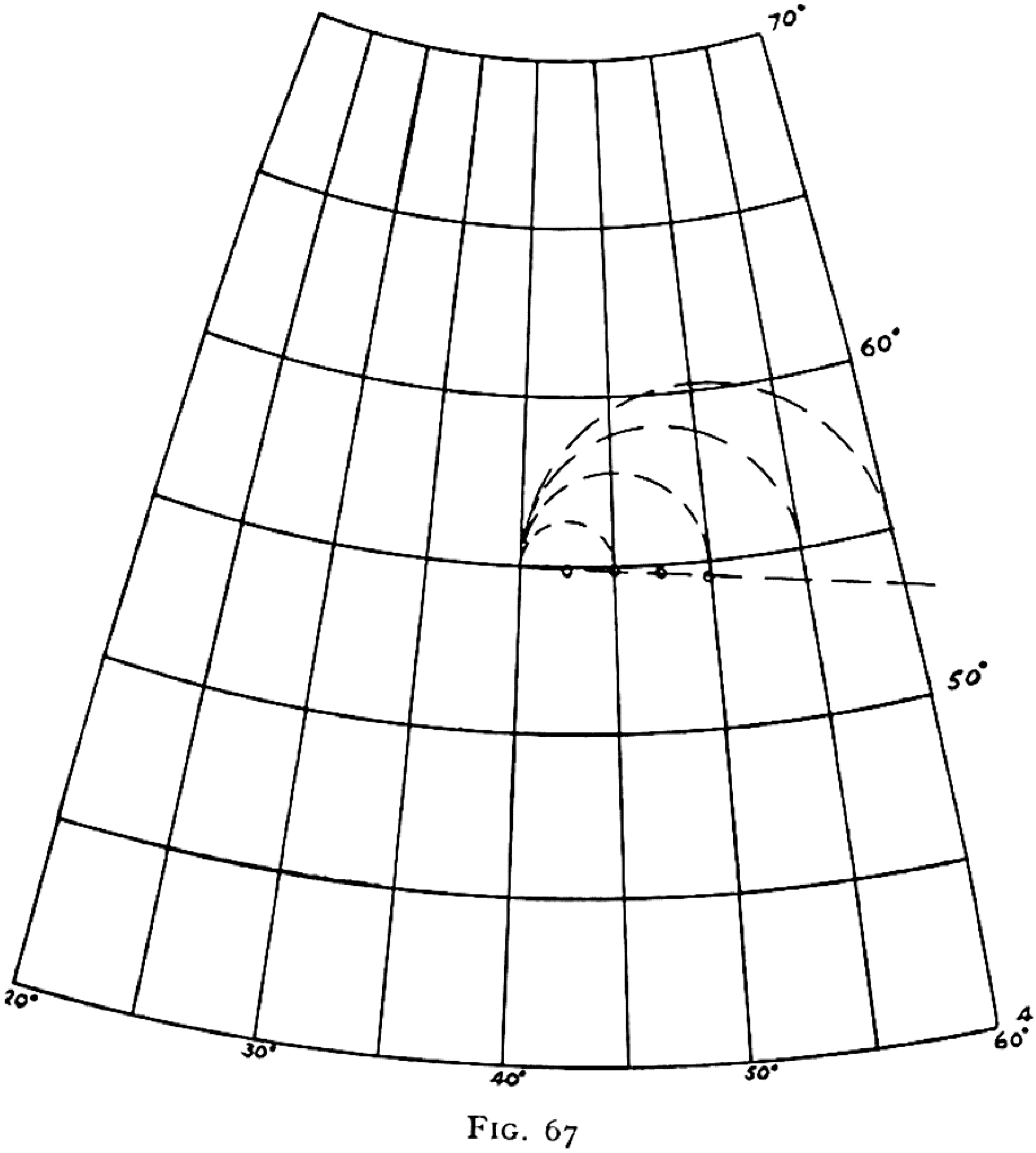


FIG. 67

arcs, and the meridians are curved lines which do not cut the parallels at right angles. In the rectangular polyconic the meridians are again modified so as to cross the parallels at exactly 90°, but this can only be done by abandoning the true scales along the parallels and, indeed, by abandoning even a uniform scale along each parallel. The scale along each parallel is true only close to the central meridian, and decreases as we move outwards from it. The construction

is exactly the same as for the ordinary polyconic until we commence to mark off the distances along each parallel in order to draw the meridians. This marking off is now done in a different way, known as "O'Farrell's Construction," which is very simple, though the proof involves mathematics beyond the scope of this book.

At each point on the central meridian through which a parallel has been drawn, draw a line perpendicular to the central meridian (Fig. 67), and along this line mark off half the true distance of each meridian from the central meridian, i.e. for meridians at 10° intervals at latitude ϕ, mark off $\frac{1}{2}R \cos \phi$ arc 10°, $\frac{1}{2}R \cos \phi$ arc 20°, etc. From the points thus found as centres describe arcs (not quite semicircles) of the same radius as the distance marked off, so that they each touch the central meridian, cutting the parallels themselves at points which, when joined up with curved lines, give the meridians. If the student has constructed an ordinary polyconic for a reduced earth of 5·73 in. radius, it will be instructive for him to modify it (in dotted lines) into a rectangular polyconic, halving the meridian intervals for each parallel given in Table I, Vol. I, and extending the range of longitude from 40° to 60° or 80°. He will note that the meridians gradually fall more and more inside those of the ordinary polyconic as he works outwards from the central meridian, indicating the reduction of the scale along the parallels. This projection has been much used by the British War Office; it is obviously suited best for a region of great extent in latitude, but of small extent in longitude.

The Conical Equal-area with One Standard Parallel (Lambert's).

The formula for the polar zenithal equal-area projection is $r = 2R \sin \frac{X}{2}$ where X is the co-latitude, and a zenithal projection is a special case of a conical projection, where n, the constant, $= 1$. The parallels are too long in the ratio $\frac{2R \sin X/2}{R \sin X} = \sec \frac{X}{2}$, and the meridian scale is too small in the ratio $\cos \frac{X}{2} : 1$. Now, if we alter the radius to r

$= m\, 2R \sin \frac{X}{2}$ and the constant to $n < 1$, we shall turn this projection into an equal-area conical projection provided we choose m and n, so that the area between the pole and any parallel remains true, also we can make any one parallel a standard parallel.

The area of a sector of angle θ of the spherical cap of co-latitude X on the reduced earth $= \frac{\theta}{2} 4R^2 \sin^2 \frac{X}{2}$, and this must equal the sector of angle $n\theta$ of radius $r = 2mR \sin \frac{X}{2}$ on the map $= \frac{n\theta}{2} 4m^2R^2 \sin^2 \frac{X}{2}$. $\therefore nm^2 = 1$. We can, therefore, obviously satisfy the second condition, that parallel of co-latitude X_0 shall be correct in length. On the reduced earth its length is $\theta R \sin X_0$; on the map its length is $n\theta r$ $= n\theta\, 2mR \sin \frac{X_0}{2}$. $\therefore \theta R \sin X_0 = n\theta\, 2mR \sin \frac{X_0}{2}$. $\therefore mn$ $= \cos \frac{X_0}{2}$. $\therefore m = \frac{1}{\cos X_0/2}$; $n = \cos^2 \frac{X_0}{2}$. Our formula for the radius of the parallels is, therefore, $r = 2R \sec \frac{X_0}{2} \cdot \sin \frac{X}{2}$.

The meridian scale at any co-latitude X will be $m \cos \frac{X}{2}$ $= \frac{\cos X/2}{\cos X_0/2}$, and the parallel scale will be

$$\frac{nr}{R \sin X} = \frac{\cos^2 X_0/2 \,.\, 2R \,.\, \sec X_0/2 \,.\, \sin X/2}{2R \sin X/2 \,.\, \cos X/2} = \frac{\cos X_0/2}{\cos X/2}$$

= reciprocal of the meridian scale. The pole will be a point as $r = 0$, when $X = 0$ and the meridian scale will continuously decrease as X increases, i.e. as the latitude decreases. The parallels will, therefore, become closer and closer together the farther we go from the pole, similarly to the polar zenithal equal-area projection from which it is derived. It, therefore, follows the general rule for equal-area projections when we remember that the *pole* is really the centre of the map, *not* a point at the average latitude.

To construct such a projection for the region from 40° – 70° N., 20° – 60° E. for a sphere 5·73 in. radius (on which 10° of latitude = 1 in.), with 55° as standard latitude. We

first calculate m and n. $m = \sec 17\frac{1}{2}° = 1{\cdot}0485$, while $n = \cos^2 17\frac{1}{2}° = {\cdot}9095$. Then $r = 2{\cdot}0970\ R \sin X/2$ and $\log r = \log \sin X/2 + 1{\cdot}0798$, while for the meridian scale s we have $s = \dfrac{\cos X/2}{\cos X_o/2}$. $\therefore \log s = \log \cos X/2 - \bar{1}{\cdot}9794$. In this way we prepare the following table—

Latitude	$X/2$	r	δr	Meridian Scale	Parallel Scale
		Inches			
70°	10°	2·086		1·033	0·968
			0·515		
65°	12½°	2·601		1·023	0·977
			0·510		
60°	15°	3·111		1·013	0·987
			0·502		
55°	17½°	3·613		1·000	1·000
			0·498		
50°	20°	4·111		·985	1·015
			0·487		
45°	22½°	4·598		·969	1·032
			0·480		
40°	25°	5·078		·950	1·052

Note that the meridian scale error ranges from $-5{\cdot}0$ per cent to $+3{\cdot}3$ per cent, and averages 2·06 per cent. To draw the projection (Fig. 68), draw a straight central meridian and set out the other meridians at angles of $n\ 10°$ and $n\ 20° = 9{\cdot}095°$ and $18{\cdot}19°$. On squared paper this is best effected by marking off 5 in. from the apex on the central meridian and $5 \times \tan 9°\ 5{\cdot}7' = {\cdot}800$ in., and $5 \times \tan 18°\ 11{\cdot}4' = 1{\cdot}643$ in. at right angles to it, and joining these points to the apex. Then describe arcs of radius r from the apex and the projection is complete. It is doubtful whether this projection is ever used in practice; it is unsuitable for any great range in latitude. Figures are elongated north and south on the polar side of the standard parallel, and east and west on the equatorial side.

Conical Equal-area with Two Standard Parallels (Albers').

This is a much better projection. Let ϕ_1 and ϕ_2 be the latitudes of the two standard parallels, say at $\frac{1}{6}$ and $\frac{5}{6}$ of the

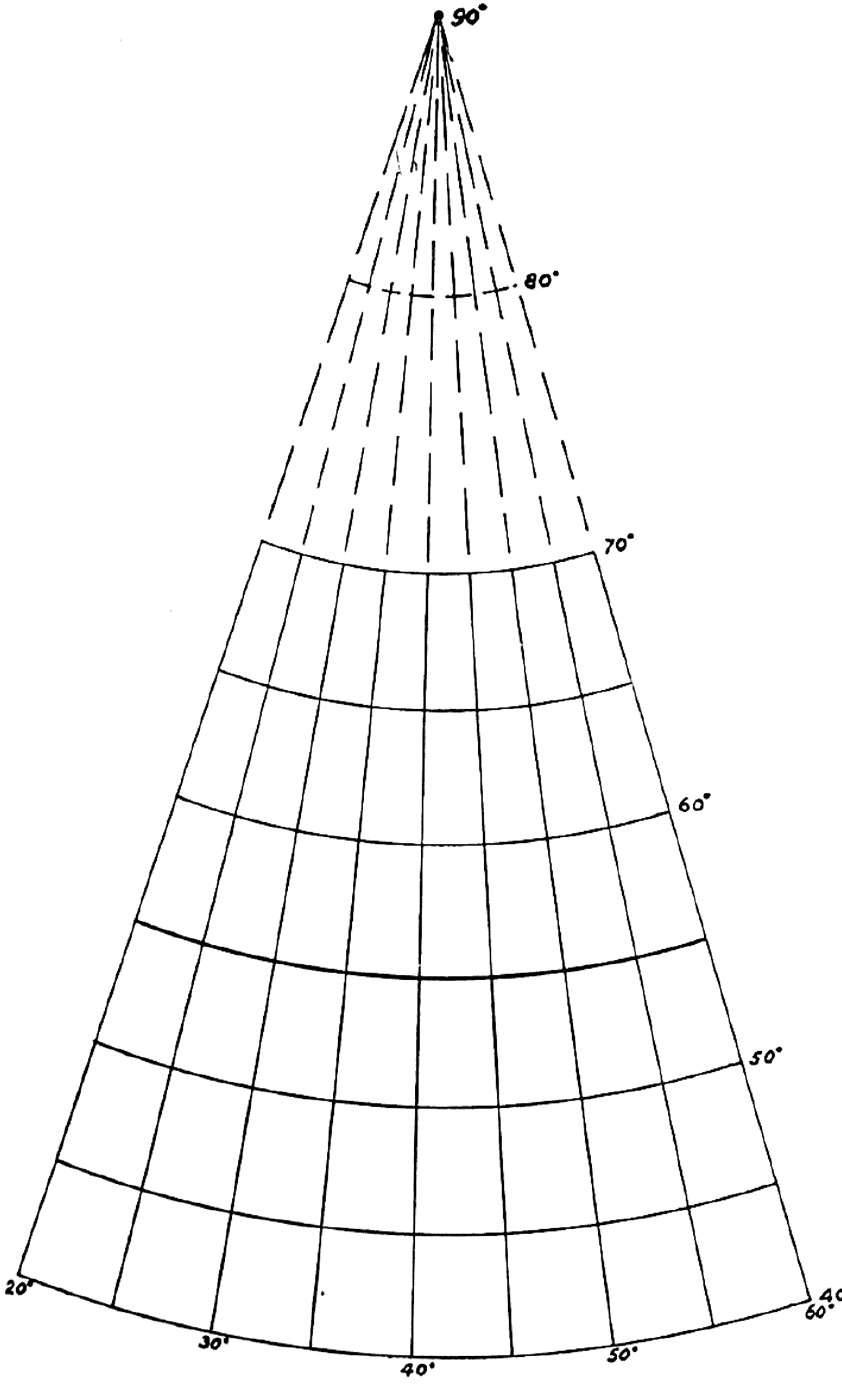


FIG. 68

latitude range. Then, as the lengths of these two parallels are to be correct, $r_1 = \frac{R \cos \phi_1}{n}$, $r_2 = \frac{R \cos \phi_2}{n}$. For the area between these two parallels to be correct we have $\frac{n}{2}(r_2{}^2 - r_1{}^2) = R^2 (\sin \phi_1 - \sin \phi_2)$. Substituting for r_2, r_1 we have $\frac{nR^2}{2} \frac{\cos^2 \phi_2 - \cos^2 \phi_1}{n^2} = R^2 (\sin \phi_1 - \sin \phi_2)$.

$$\therefore n = \frac{\cos^2 \phi_2 - \cos^2 \phi_1}{2(\sin \phi_1 - \sin \phi_2)} = \frac{\sin^2 \phi_1 - \sin^2 \phi_2}{2(\sin \phi_1 - \sin \phi_2)} = \frac{\sin \phi_1 + \sin \phi_2}{2}$$

= constant of the cone. Lastly, for the area between any parallel ϕ and the standard parallel ϕ_2 to be correct, we have $\frac{n}{2}(r_2{}^2 - r^2) = R^2(\sin \phi - \sin \phi_2)$.

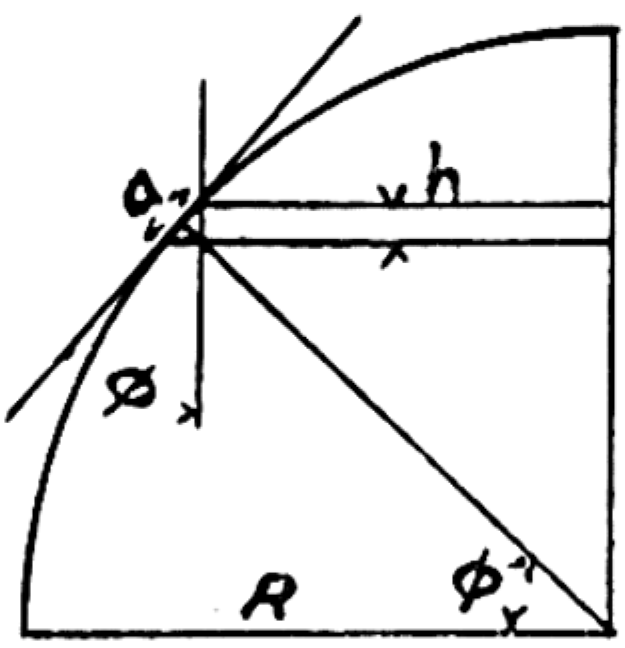


FIG. 69

$$\therefore \quad r_2{}^2 - r^2 = \frac{2R^2(\sin \phi - \sin \phi_2)}{n}$$

$$\therefore r^2 = r_2{}^2 - \frac{2R^2(\sin \phi - \sin \phi_2)}{n}$$

which is the formula for the radii of the parallels. The pole is *not* a point, as its radius on the map is $\sqrt{r_2{}^2 - \frac{2R^2(1 - \sin \phi_2)}{n}}$ which is not zero.

The parallel scale $= \frac{nr}{R \cos \phi}$, and the meridian scale s must be the reciprocal of this as we have made the projection equal area, i.e. $\frac{R \cos \phi}{nr}$, but we can verify this as follows: Consider two parallels ϕ and ϕ' very close together, then we have, if r, r' are their radii on the map, $r^2 = r_2{}^2 - \frac{2R^2(\sin \phi - \sin \phi_2)}{n}$, $r'^2 = r_2{}^2 - \frac{2R^2(\sin \phi' - \sin \phi_2)}{n}$.

$$\therefore r^2 - r'^2 = \frac{2R^2}{n} (\sin \phi' - \sin \phi) = \frac{2R}{n} (R \sin \phi' - R \sin \phi)$$

$= \frac{2R}{n} \times$ height (h) of zone, between parallels, $\perp$ equator (Fig. 69).

But the meridian scale,

$$s = \frac{r - r'}{a} \text{ where } a \text{ is the true meridian distance between the parallels}$$

$$= \frac{r^2 - r'^2}{(r + r')a} = \frac{r^2 - r'^2}{2ra} \text{ very nearly} = \frac{2R}{n} \times \frac{h}{2ra} \text{(as above)}$$

$$= \frac{R \cos \phi}{nr} \text{ as stated above.}$$

To compute the projection for the region $40° - 70°$ N., $20° - 60°$ E. for a reduced earth 5·73 in. radius, the standard parallels being $\phi_1 = 65°$, $\phi_2 = 45°$, we have $n = \frac{\sin 45° + \sin 65°}{2}$ $= \cdot 8067$. $r_2 = \frac{5{\cdot}73 \cos 45°}{n} = 5{\cdot}022$ in., and our formula is $r^2 = 25{\cdot}22 - \frac{2 \times 5{\cdot}73^2}{\cdot 8067}(\sin \phi - \sin 45°) = 25{\cdot}22 - 81{\cdot}39 (\sin \phi - \cdot 7071)$.

We prepare the following table—

ϕ	r	δr	Meridian Scale	Parallel Scale
	Inches			
70°	2·508		0·969	1·032
		0·494		
65°	3·002		1·000	1·000
		0·503		
60°	3·505		1·013	0·987
		0·507		
55°	4·012		1·016	0·984
		0·508		
50°	4·520		1·010	0·990
		0·502		
45°	5·022		1·000	1·000
		0·496		
40°	5·518		0·986	1·014

We note how little δr, i.e. the increment of r, varies from its true value, ·500 in., for 5°, also that the meridian scale

errors range from $-3\cdot1$ per cent to $+1\cdot6$ per cent and average only $1\cdot02$ per cent, and these might be reduced by selecting the standard parallels more judiciously as in the Simple Conical at the commencement of this chapter. The errors, as it is, are only about half those in the last projection. We draw this (Fig. 70) in a similar way to the last projection, setting off at a distance 5 in. from the apex $5 \times \tan 8\cdot067^\circ = \cdot708$ in. and $5 \times \tan 16\cdot134^\circ = 1\cdot446$ in. for the meridians, and describing arcs of radius r. It will be noticed that it conforms to the general rule for equal-area projections, viz., that the parallels become closer together the farther the centre (55°) is departed from. This is a very good projection, and has been used for both Austria and Russia.

The Conical Orthomorphic with Two Standard Parallels (Gauss').

It can be shown by the use of the Differential Calculus[1] that if the radii of the parallels are made $r = m\left(\tan\frac{X}{2}\right)^n$ where m is a scale constant and n is the constant of the cone, chosen so as to make the two selected standard parallels of true length, the projection will be orthomorphic. The parallel scale is, of course, $\frac{nr}{R\sin X}$ where X is the co-latitude, and the meridian scale will be the same.

To find m and n: for the lengths of the standard parallels, X_1 and X_2, to be correct, $nr_1 = R\sin X_1$, $nr_2 = R\sin X_2$.

$$\therefore r_1 = \frac{R\sin X_1}{n};\ r_2 = \frac{R\sin X_2}{n}$$

$$\therefore \frac{\sin X_2}{\sin X_1} = \frac{r_2}{r_1} = \frac{m(\tan X_2/2)^n}{m(\tan X_1/2)^n} = \frac{(\tan X_2/2)^n}{(\tan X_1/2)^n}$$

$$\therefore \log\sin X_2 - \log\sin X_1 = n\left(\log\tan\frac{X_2}{2} - \log\tan\frac{X_1}{2}\right)$$

[1] Students who are not conversant with the calculus should calculate the meridian scale at, say, 40° approximately by calculating r for $40\tfrac{1}{2}^\circ$ and $39\tfrac{1}{2}^\circ$, and find the increment for 1°. This, divided by $\cdot100$ in., its true value, is the meridian scale at 40° very nearly. (See note at end of chapter.)

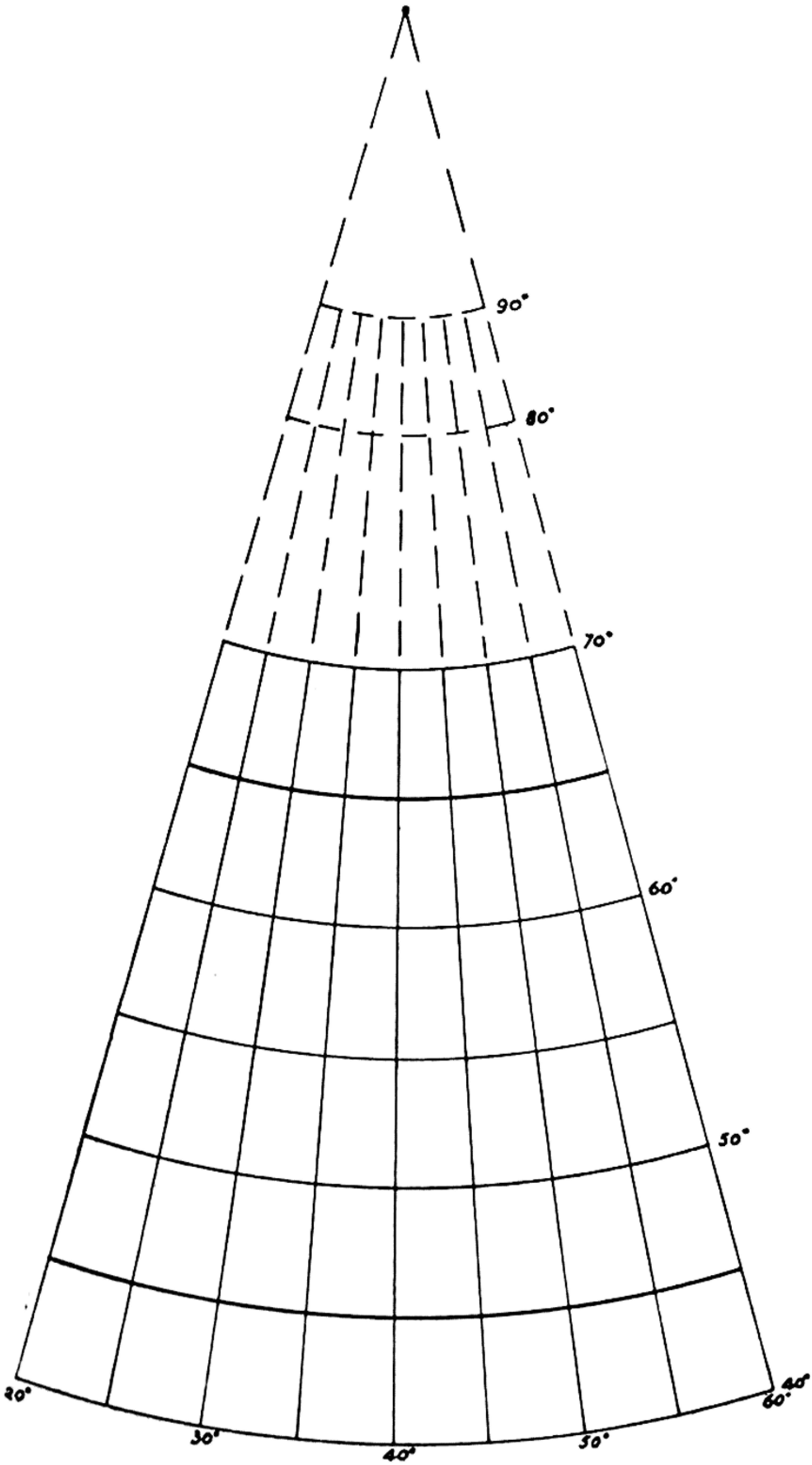


FIG. 70

$$\therefore \quad n = \frac{\log \sin X_2 - \log \sin X_1}{\log \tan X_2/2 - \log \tan X_1/2}$$

$$\text{while } m = \frac{r_2}{(\tan X_2/2)^n} = \frac{R \sin X_2}{n(\tan X_2/2)^n}$$

The pole will be a point, as $r = 0$, when $X = 0$; at the equator where $X = 90°$ the radius, r, of the parallel will be m.

To compute the projection for the region 40° – 70° N., 20° E. to 60° E. for a reduced earth of radius 5·73 in., with 45° and 65° as our standard parallels. We first find $n = \frac{\log \sin 45° - \log \sin 25°}{\log \tan 22\frac{1}{2}° - \log \tan 12\frac{1}{2}°} = \cdot 8234$, while $m = \frac{5\cdot 73 \sin 45°}{\cdot 8234 (\tan 22\frac{1}{2}°)^n}$ $= 10\cdot 167$ in. $\therefore \log r = \log 10\cdot 167 + \cdot 8234 \log \tan \frac{X}{2} =$ $1\cdot 0072 + \cdot 8234 \log \tan \frac{X}{2}$, and we calculate the following table—

Latitude	$\frac{X}{2}$	r	δr	Meridian and Parallel Scale
		Inches		
70°	10°	2·435		1·023
			·506	
65°	12½°	2·941		1·000
			·497	
60°	15°	3·438		0·988
			·492	
55°	17½°	3·930		0·984
			·494	
50°	20°	4·424		0·989
			·496	
45°	22½°	4·920		1·000
			·505	
40°	25°	5·425		1·018

Again, we notice how little the increments (δr) differ from their true value, ·500 in., for 5°, also that the scale errors range from – 1·6 per cent to + 2·3 per cent and average 0·99 per cent, even less than those in Alber's projection, and these might be reduced by a more suitable choice of standard parallels. The student should plot this projection (Fig. 71),

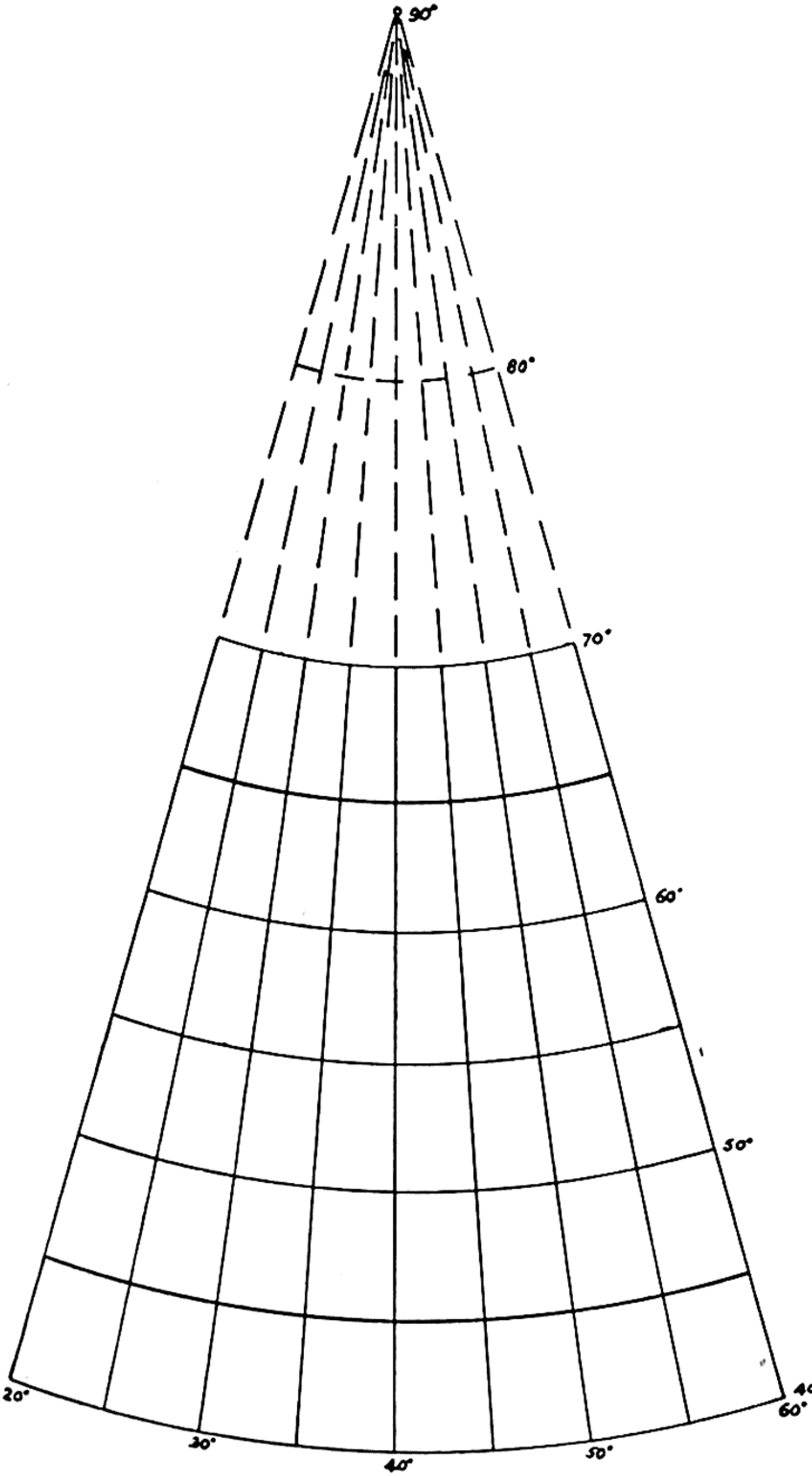


FIG. 71

setting out 5 in. along the central meridian and at right angles distances of $5 \times \tan 8{\cdot}234°$ and $5 \times \tan 16{\cdot}468°$. It conforms to the general rule for orthomorphic projections that the parallels become farther apart as we depart from the centre (55°) of the map. This is a very good projection, indeed, and has been used for Russia, Europe, and Australia ; if the range in latitude is not too great, the small scale errors keep the errors in area comparatively small.

$$\left[\text{Note: Meridian scale} = \frac{dr}{Rdx} = \frac{mn}{2R}\left(\tan\frac{X}{2}\right)^{n-1}\sec^2\frac{X}{2}\right.$$

$$\left. = \frac{nr}{2R\tan X/2 \,.\, \cos^2 X/2} = \frac{nr}{2R\sin X/2\cos X/2} = \frac{nr}{R\sin X}\right]$$

CHAPTER IX

EFFECT OF THE SPHEROIDAL SHAPE OF THE EARTH ON MAP PROJECTIONS

The Spheroidal Shape of the Earth.

It is shown in the first part of this Volume that the earth is not quite a sphere, but is an ellipsoid of revolution, or

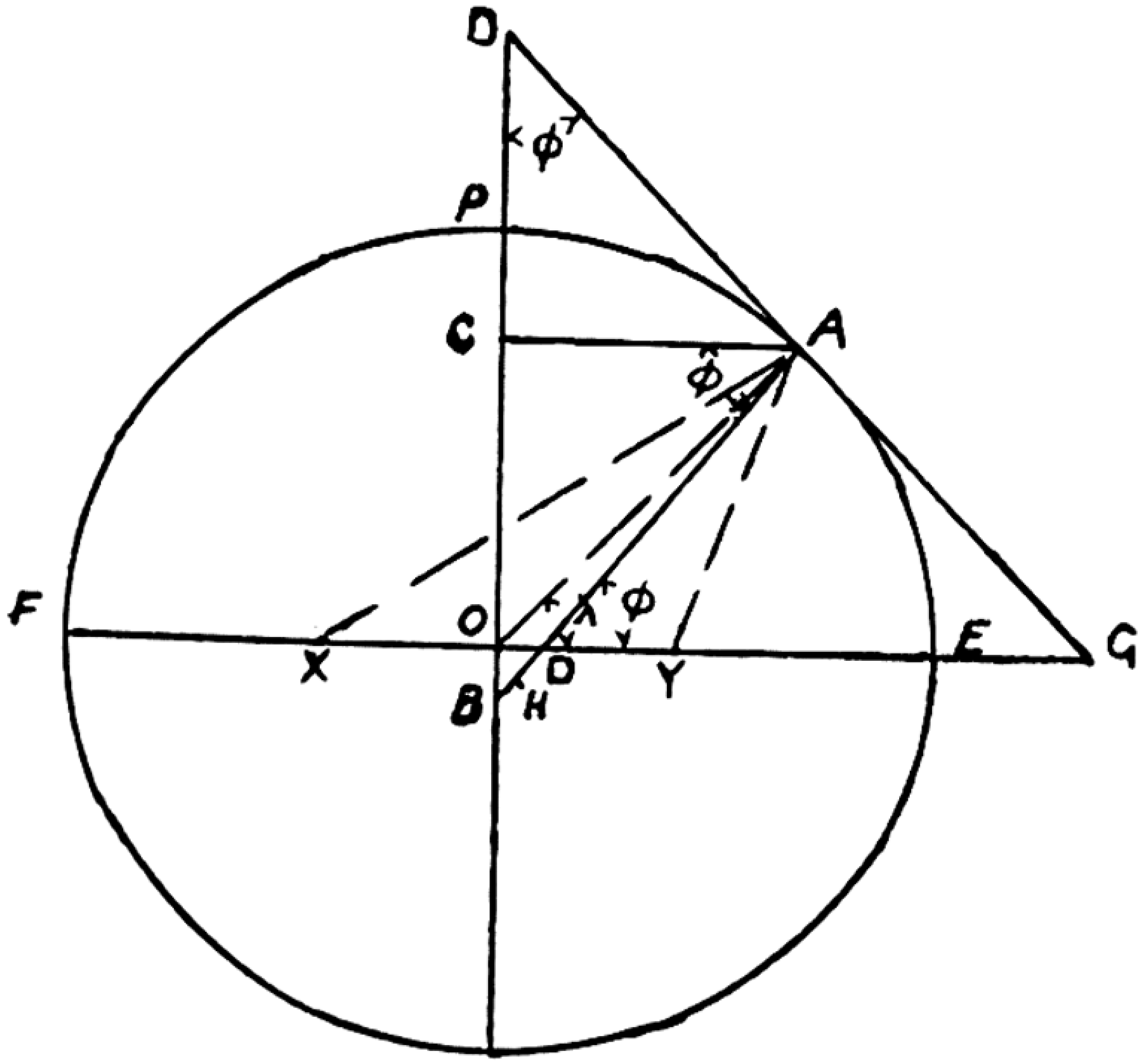


FIG. 72

oblate spheroid, which is the solid generated by an ellipse rotating about its minor axis. If a is the major semi-axis and b the minor semi-axis of the ellipse, the "compression" $\frac{a-b}{a} = \frac{1}{297}$, so that the polar axis is $\frac{296}{297}$ of the equatorial axis. If from the pole (Fig. 72) or end of the minor axis we strike an arc of radius a, cutting the major axis at X, Y, these points are the "foci," such that the sum of their distances from any point A on the ellipse $= 2a$. The radius joining O, the centre of the ellipse, to any point A is not now constant

in length, but can be shown to be equal to $\dfrac{ab}{\sqrt{a^2 \sin^2 \lambda + b^2 \cos^2 \lambda}}$ where λ is the angle made by OA with the major axis, called the "geocentric latitude." The tangent at A is no longer perpendicular to the radius OA, but is perpendicular to the "normal" AB which bisects the angle XAY. The angle which the normal makes with the major axis is called the "geographical latitude" ϕ, connected with λ by the relation $\tan \lambda = \dfrac{b^2}{a^2} \tan \phi$. The "centre of curvature" of the elliptical meridian at A lies at H on AB between B and D, and AH is the "radius of curvature" of the meridian for a short distance; we shall call this ρ. B is the "centre of curvature" at right angles to the meridian, AB the "radius of curvature" at right angles to the meridian for a short distance: we shall call this ν. The value of ρ increases from $\dfrac{b^2}{a}$ at the equator to $\dfrac{a^2}{b}$ at the poles—an increase of about 1 per cent. The value of ν also increases from a at the equator to $\dfrac{a^2}{b}$ at the poles—an increase of about $\frac{1}{3}$ per cent. The radius of the parallel of latitude at A is $AB \cos \phi$ or $\nu \cos \phi$; that of the equator is a.

The most convenient way to express these variations of the two radii of curvature of the surface for the cartographer is by tables giving the lengths in miles (or kilometres) of 1° of latitude and of longitude at different geographical latitudes as in Table I, taken from Hink's *Map Projections*, the values at the intermediate 5° being interpolated graphically.

TABLE I

Latitude	1° of Meridian	1° of Parallel	Latitude	1° of Meridian	1° of Parallel
	Miles	Miles		Miles	Miles
0°	68·70	69·17	50°	69·12	44·55
5°	68·71	68·91	55°	69·18	39·77
10°	68·73	68·13	60°	69·23	34·67
15°	68·76	66·83	65°	69·28	29·31
20°	68·79	65·03	70°	69·32	23·73
25°	68·83	62·73	75°	69·36	17·96
30°	68·88	59·96	80°	69 39	12·05
35°	68·93	56·73	85°	69·40	6·05
40°	68·99	53·06	90°	69·41	0·00
45°	69·06	49·00			

(The lengths of 1° of meridian are for arcs extending from ½° South to ½° North of the latitude.)

The length of 1° of meridian is, of course, $\frac{\rho \text{ miles}}{57{\cdot}30}$, that of 1° of parallel is $\frac{\nu \text{ miles} \times \cos \phi}{57{\cdot}30}$. It will be noticed that 1° of meridian (or latitude) at the equator is shorter than 1° of longitude at the equator. If the earth were a sphere of the same radius as the equator, 1° of longitude at 50° would be 44·46 miles.

We shall now consider the effect of this spheroidal shape on the various map projections.

The Simple Cylindrical Projection.

Suppose we wished to draw this projection to a scale of 500 miles to an inch from 0° to 30° N. It would be a very bad projection, but it forms a good illustration. We draw a straight line for the equator, and mark off along it $\frac{69{\cdot}17 \times 5}{500} = {\cdot}6917$ in. for each 5° of longitude required, and through these points draw lines perpendicular to the equator for the meridians. Then, if we had a more complete table on the lines of Table I, but giving the lengths of 1° of latitude for every 1° (or even closer increments) of latitude, we should find the distance, y, of each parallel of latitude from the equator by summing up the lengths of 1° of meridian up to the latitude in question (taking, however, only one half of the first and last lengths). But for such a small scale map the method of "averaging" given at the foot of page 144 is sufficiently correct.

Then we draw lines parallel to the equator at these distances, y, from it to complete the projection. The scale along the meridians would be correct, but that on the 10° parallel (say) would be too large in the ratio $\frac{69{\cdot}17}{68{\cdot}13}$, i.e. in the ratio $\frac{\text{length of 1° of longitude at 0°}}{\text{length of 1° of longitude at 10°}}$.

The Cylindrical Equal-area Projection.

We should draw the meridians exactly as before, but for the parallels we must *reduce* the length of each 1°

of latitude taken from the *complete* tables by multiplying it by

$$\frac{\text{length of } 1^\circ \text{ of longitude at the latitude in question}}{\text{length of } 1^\circ \text{ of longitude at } 0^\circ},$$

e.g. at 10° latitude we must make our increment for 1° of latitude, $68{\cdot}73 \times \frac{68{\cdot}13}{69{\cdot}17} = 67{\cdot}70$ miles, and then sum up all these 1° increments of y to obtain the distance, y, for each parallel from the equator. This reduction of the meridian scale at each latitude compensates for the exaggeration of the parallel scale explained in the last paragraph.

The Cylindrical Orthomorphic (Mercator's) Projection.

Again we draw the meridians as before, but the length of 1° of meridian in the *complete* tables must be *enlarged* in the ratio of

$$\frac{\text{length of } 1^\circ \text{ of longitude at } 0^\circ}{\text{length of } 1^\circ \text{ of longitude at the latitude in question}},$$

e.g. at 10° latitude we must make our increment for 1° of latitude, $68{\cdot}73 \times \frac{69{\cdot}17}{68{\cdot}13} = 69{\cdot}78$ miles, and then sum these increments up to the latitude in question to obtain the distance, y,

TABLE FOR SIMPLE CYLINDRICAL PROJECTION

Latitude	1° of Meridian	Average	Length of 5° on Map	Distance from Equator, y
	Miles	Miles	Inches	Inches
0°	68·70			0·000
		68·705	·68705	
5°	68·71			0·68705
		68·72	·6872	
10°	68·73			1·37425
		68·745	·68745	
15°	68·76			2·06170
		68·775	·68775	
20°	68·79			2·74945
		68·81	·6881	
25°	68·83			3·43755
		68·855	·68855	
30°	68·88			4·12610

of that parallel from the equator. It should be noticed that the first few degrees of latitude will only be slightly enlarged, so that they will still remain shorter than degrees along the equator. Consequently, the distance from the equator to the 10° parallel will be slightly *shorter* than the length of 10° of longitude on our map, e.g. at 5° latitude 1° of latitude will be enlarged to $68{\cdot}71 \times \frac{69{\cdot}17}{68{\cdot}91} = 68{\cdot}97$ miles, which is less than 69·17 miles. This laborious "step by step" method would not be accurate unless the "steps" were much smaller than 1° of latitude, nor would it be employed in practice. Just as for a sphere, the Integral Calculus gives the distance of the parallel ϕ from the equator as $y = R \log_e \tan \frac{90^\circ + \phi}{2}$, so for a spheroid the formula is $y = a\left(\log_e \cdot \tan \frac{90^\circ + \phi}{2} - e^2 \sin \phi - \frac{e^4 \sin^3 \phi}{3} - \ldots\right)$, where $e^2 = \frac{a^2 - b^2}{a^2}$, e being the "eccentricity" of the ellipse. (See Note on p. 153.)

The Sinusoidal Projection.

This is readily constructed from Table I. The central meridian is drawn as a straight line, and the true distances of the parallels, north and south of the equator, are computed exactly as for the simple cylindrical, and straight lines are drawn at right angles to the central meridian through the points thus found. Along each parallel mark off the intervals for 5° of longitude for that latitude, e.g. at 10° latitude mark off $\frac{68{\cdot}13 \times 5}{500} = {\cdot}6813$ in. both ways from the central meridian as often as required, and through these points draw the curved meridians.

The Simple Conical with One Standard Parallel.

Here we must first find the radius of the standard parallel ϕ_0 on the earth, viz., AC in Fig. 72. This is equal to the length of 1° of the standard parallel × 57·30. The radius of the standard parallel on the map is then $AD = AC \,.\, \text{cosec}\, \phi_o$. The distances along the straight central meridian must be computed exactly

as for the simple cylindrical. As an example we shall compute the projection for the region 40° – 70° N., 20° E. to 60° E., with standard parallel 55° N., to a scale of 500 miles to 1 in. From Table I we find the length of 1° of longitude at 55° latitude is 39·77 miles. Therefore, the radius of the parallel on the earth is 57·30 × 39·77 miles = 2,279 miles = 4·558 in. to our scale. The radius of this parallel on our map will be 4·558 × cosec 55° = 5·564 in. We describe an arc of this radius, and through its centre we draw a straight central meridian, and on each side of it along the arc mark off four times the length of 5° of this parallel, viz., $\frac{39·77 \times 5}{500} = ·3977$ in. We join these points to the centre of the arc with straight lines, giving the meridians at 5° intervals. For the parallel intervals we make a table—

Latitude	1° of Meridian	Average	Length of 5° on Map	Distance from 55° Parallel	Radius of Parallel
	Miles	Miles	Inches	Inches	Inches
70°	69·32				3·486
		69·30	·6930	2·0776	
65°	69·28				4·179
		69·255	·69255	1·3846	
60°	69·23				4·872
		69·205	·69205	·69205	
55°	69·18				5·564
		69·15	·6915	·6915	
50°	69·12				6·255
		69·09	·6909	1·3824	
45°	69·06				6·946
		69·025	·69025	2·07265	
40°	68·99				7·637

We mark off these distances up and down from the standard parallel, and describe concentric circles through them. The scale is, however, too small for the effect of the earth's figure to be really appreciable. The 1° lengths on the 70°, 55°, and 40° parallels on the map will be proportioned to the radii, i.e. will be $\frac{3·486}{5·564} \times 39·77$, $39·77$, and $\frac{7·637}{5·564} \times 39·77$ miles to

our scale, i.e. 24·92, 39·77, 54·59 miles as compared with their true lengths, 23·73, 39·77, and 53·06 miles, i.e. the errors are + 5·0 per cent, 0 per cent, and + 2·9 per cent.

Bonne's Projection.

This would be constructed in the same way as the last projection, except that the true meridian intervals $\left(\text{in our case, } \frac{\text{length of } 1^\circ \text{ of parallel} \times 5}{500}\right)$ must be marked off along each parallel from Table I, and the points so found joined by curved meridians.

The Polyconic.

Here we must calculate first the radius of *each* parallel on the map as if it were the standard parallel in the simple conical with one standard parallel, i.e. we must calculate length of 1° of parallel × 57·30 × cosecant of the latitude, mark off the true intervals between the parallels along the central meridian as the last table, and through each of the points thus found describe an arc of the radius calculated for that latitude—these circles, of course, being non-concentric, but all their centres being on the central meridian. Then, as in Bonne's, mark off the true meridian intervals along each parallel and join these points with curved meridians.

The Simple Conical with Two Standard Parallels.

This is the easiest conical to construct, and, as it is much more accurate than that with one standard parallel, it should always be used in preference. The elliptical profile does not affect our calculation of radii. As before, we calculate the true distance between the standard parallels—in this case from 45° to 65° = 1·3846 + 1·3824 = 2·7670 in. Then from Table I we find that 1° of parallel at 45° is 49·00 miles, and at 65° is 29·31 miles.

The radius on the map of the 65° parallel is then $\frac{29{\cdot}31}{49{\cdot}00 - 29{\cdot}31} \times 2{\cdot}767 = 4{\cdot}119$ in. We then describe a circle of this radius with its centre on our straight central meridian, and mark off the calculated intervals from the last table, for 70° above, and for the other parallels below, this parallel and describe

concentric arcs through these points. Then along the 65° parallel mark off $29{\cdot}31 \times \frac{5}{500} = {\cdot}2931$ in. four times on each side of the central meridian, and along the 45° parallel $49{\cdot}00 \times \frac{5}{500} = {\cdot}4900$ in. similarly, and join these points to the centre of the circles with straight meridians. As the radii of the 70°, 55°, and 40° parallels on the map are 3·426, 5·504, and 7·576 in. respectively, 1° at these latitudes will scale $\frac{3{\cdot}426}{4{\cdot}119} \times 29{\cdot}31 = 24{\cdot}38$ miles, $\frac{5{\cdot}504}{6{\cdot}886} \times 49{\cdot}00 = 39{\cdot}17$ miles, and $\frac{7{\cdot}576}{6{\cdot}886} \times 49{\cdot}00 = 53{\cdot}91$ miles respectively, which compared with the values 23·73, 39·77, and 53·06 show errors of + 2·7 per cent, – 1·5 per cent, and + 1·6 per cent, but these could have been equalized and reduced by a better choice of standard parallels, as shown in Chapter VIII.

The Zenithal Equidistant Polar Projection.

The Zenithal equidistant polar projection is also very easily drawn. A table is prepared, and we describe circles with these distances as radii, the meridians being straight radiating lines at true angles apart.

Latitude	1° of Meridian	Average	Length of 5° on Map	Distance from Pole
	Miles	Miles	Inches	Inches
90°	69·41			0·0000
		69·405	·69405	
85°	69·40			0·69405
		69·395	·69395	
80°	69·39			1·38800
		69·375	·69375	
75°	69·36			2·08175
		69·34	·69340	
70°	69·32			2·77515
		69·30	·69300	
65°	69·28			3·46815
		69·255	·69255	
60°	69·23			4·16070

The error on the 60° parallel is found as follows: 1° of map scales $\dfrac{4 \cdot 1607 \times 500}{57 \cdot 30} = 36 \cdot 31$ miles as against the figure in Table I of 34·67 miles. Therefore, error in parallel $= + 4 \cdot 7$ per cent.

The Gnomonic (or Central) Projection.

The Gnomonic (or central) projection can also readily be drawn as a polar or equatorial projection (and the cubic projection, also) by the formulae given for a sphere if, for ϕ, the geographical latitude, we substitute λ the geocentric latitude, where $\tan \lambda = \cdot 99315 \tan \phi$, and substitute b for R in a polar projection and a for R in an equatorial one. The oblique gnomonic can also be drawn, but the formulae require more correction. Let the plane of projection touch the spheroid at C of geographic latitude ϕ_o, and geocentric latitude λ_o (Fig. 73).

$$\text{Then } OC = \frac{ab}{\sqrt{a^2 \sin^2 \lambda_o + b^2 \cos^2 \lambda_o}} = R_o \text{ (say).}$$

$$\text{Then } SC = \frac{R_o \sin (90^\circ - \lambda_o)}{\sin \phi_o}; \; TC = \frac{R_o \sin \lambda_o}{\sin (90^\circ - \phi_o)} = \frac{R_o \sin \lambda_o}{\cos \phi_o}$$

$$OT = \frac{R_o \sin (90^\circ - \lambda_o + \phi_o)}{\sin (90^\circ - \phi_o)} = R_o \frac{\cos (\phi_o - \lambda_o)}{\cos \phi_o}$$

$$OA = OT \sec \theta = \frac{R_o \cos (\phi_o - \lambda_o)}{\cos \phi_o} \sec \theta,$$

$$AT = OT \tan \theta = R_o \frac{\cos (\phi_o - \lambda_o)}{\cos \phi_o} \tan \theta,$$

$$\text{and, if } \beta = \text{angle } SAO; \tan \beta = \frac{SO}{OA} = \frac{SO}{OT \sec \theta} = \cot \phi_o \cos \theta.$$

Then, if λ be the geocentric latitude of any parallel,

$$AF = \frac{OA \sin \lambda}{\sin (\beta + \lambda)} = R_o \frac{\cos (\phi_o - \lambda_o)}{\cos \phi_o} \sec \theta \frac{\sin \lambda}{\sin (\beta + \lambda)}$$

The construction (Fig. 73) is similar to that for the sphere (Fig. 60), and the same letters have been used, but in Fig. 73 the geometrical construction is indicated, whereas Fig. 60 was partly perspective. It will be noted that all the formulae

reduce to that for the sphere, when $\lambda = \phi$, $\lambda_0 = \phi_0$, and $R_0 = R$. The shortest distance between two points on a spheroid (or the "geodesic line") is not quite the same as

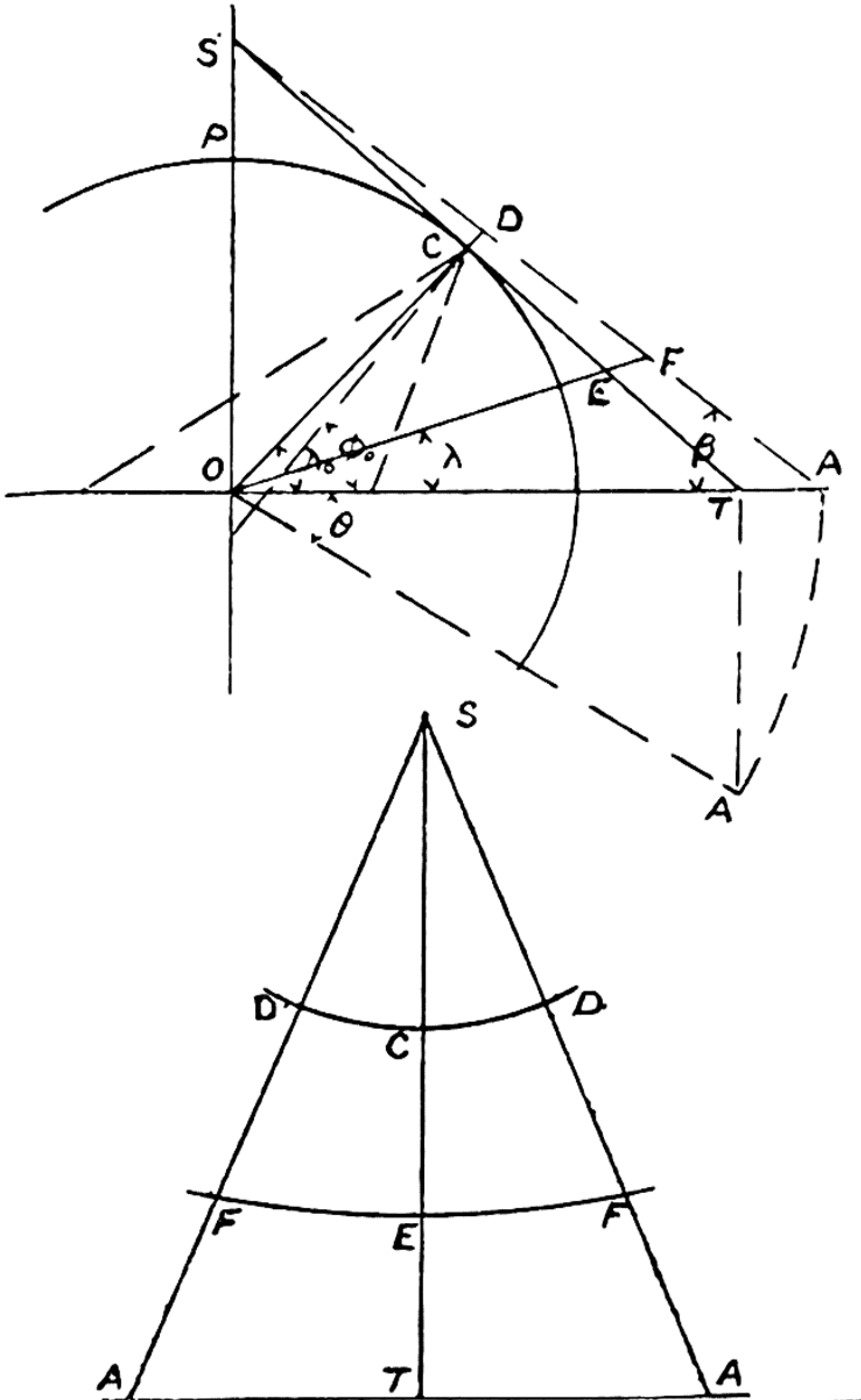


FIG. 73

the intersection of the spheroidal surface by a plane through the centre as in a sphere, but the difference is very small as the earth is so nearly a sphere.

The conical equal-area and orthomorphic projections for a

spheroid require much more advanced mathematics than the scope of this book allows, as might be gathered from the complicated formulae given for Mercator's projection for a spheroid, which, being a cylindrical projection, is of the simplest type. The zenithal equal-area and orthomorphic projections also require very advanced mathematics, further complicated by the fact that the calculation of distance and azimuth from the centre of the map on the spheroid are much more difficult than on a sphere. For small scale maps, such as atlas maps, an approximation sufficiently close is to project them from a sphere of a radius equal to the average radius of curvature over the region to be mapped. The radius of curvature, ρ, in the meridian at any latitude is length of $1°$ of latitude $\times$ 57·30, while the radius of curvature, ν, perpendicular to the meridian is length of $1°$ of longitude $\times$ 57·30 $\times$ secant of the geographic latitude. For a zenithal equal-area projection of the polar regions, $90° - 60°$ latitude, if we take $75°$ as mean latitude, we get $\rho = 69{\cdot}36 \times 57{\cdot}30$ miles, $\nu = 17{\cdot}96 \times \sec 75° \times 57{\cdot}30$. Therefore, mean radius of curvature may be taken as $\dfrac{69{\cdot}36 + 69{\cdot}39}{2} \times 57{\cdot}30 = 3{,}975$ miles for this region.

NOTES

Note on Solution of Oblique Spherical Triangles (p. 113)

Note to Chapter VII. Drop a perpendicular CM to great circle BA (Fig. 56).

Then

$$\cos C = \cos(BCM - ACM) = \cos BCM \cos ACM + \sin BCM \sin ACM$$

$$= \cos BM \sin B \cos AM \sin A + \frac{\sin BM \sin AM}{\sin a \sin b}$$

$$= \frac{\sin a \sin B \sin b \sin A \cos BM \cos AM + \sin BM \sin AM}{\sin a \sin b}$$

$$= \frac{\sin^2 CM \cos BM \cos AM + \sin BM \sin AM}{\sin a \sin b}$$

$$= \frac{\cos(BM - AM) - \cos CM \cos BM \cos CM \cos AM}{\sin a \sin b}$$

$$= \frac{\cos c - \cos a \cos b}{\sin a \sin b}$$

$$\therefore \underline{\underline{\cos c = \cos a \cos b + \sin a \sin b \cos C.}}$$

$$\therefore \quad \cos X = \sin \phi_o \sin \phi + \cos \phi_o \cos \phi \cos \theta.$$

Also $\sin CM = \sin b \sin A = \sin a \sin B$. $\therefore \dfrac{\sin A}{\sin a} = \dfrac{\sin B}{\sin b}$

Similarly it may be proved that

$$\frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}. \quad \therefore \sin B = \sin \alpha = \frac{\cos \phi \sin \theta}{\sin X}$$

Note on Mercator's Projection of Spheroid (p. 145)

Note to Chapter IX. From the properties of an ellipse it can be shown that the radii of curvature of the spheroid are are $\rho = \dfrac{(1 - e^2)a}{(1 - e^2 \sin^2 \phi)^{\frac{3}{2}}}$; $\nu = \dfrac{a}{(1 - e^2 \sin^2 \phi)^{\frac{1}{2}}}$. On the earth the

length of an increment $d\phi$ of latitude is $\rho \,.\, d\phi$, while the length of a difference, θ, of longitude at latitude ϕ is $\theta\nu \,.\, \cos\phi$. At the equator the length of this longitude difference is $a\theta$. Therefore, the increment of latitude on the map must be $dy = \rho\, d\phi \dfrac{a\theta}{\theta\nu\cos\phi} = a\dfrac{1-e^2}{1-e^2\sin^2\phi}\dfrac{d\phi}{\cos\phi}$ and the distance of the parallel from the equator is

$$y = a(1-e^2)\int_0^\phi \frac{d\phi}{(1-e^2\sin^2\phi)\cos\phi}$$

$$= (1-e^2)\int_0^\phi \frac{\cos\phi\, d\phi}{(1-\sin^2\phi)(1-e^2\sin^2\phi)}$$

(Put $x = \sin\phi$; $dx = \cos\phi \,.\, d\phi$)

$$= a(1-e^2)\int_0^x \frac{dx}{(1-x^2)(1-e^2x^2)}$$

$$= \frac{a(1-e^2)}{1-e^2}\int_0^x \left(\frac{1}{1-x^2} - \frac{e^2}{1-e^2x^2}\right) dx$$

$$= a\int_0^x \frac{dx}{1-x^2} - ae^2\int_0^x \frac{dx}{1-e^2x^2}$$

$$= a\left\{\tfrac{1}{2}\int_0 \left(\frac{1}{1-x} + \frac{1}{1+x}\right) dx - e^2\int_0^x (1-e^2x^2)^{-1}\, dx\right\}$$

$$= a\left\{\tfrac{1}{2}\log_e \frac{1+x}{1-x} - e^2\int_0^x (1 + e^2x^2 + e^4x^4 + \quad . \;.)\, dx\right\}$$

$$= a\left\{\tfrac{1}{2}\log_e \frac{1+\sin\phi}{1-\sin\phi} - e^2\left(x + e^2\frac{x^3}{3} + e^4\frac{x^5}{5} + \,.\;.\right)\right\}$$

$$= a\left\{\log_e \sqrt{\frac{1+\sin\phi}{1-\sin\phi}} - e^2\sin\phi - \frac{e^4\sin^3\phi}{3} - \;.\;.\;.\right\}$$

$$= a\left\{\log_e \tan\frac{90^\circ+\phi}{2} - e^2\sin\phi - \frac{e^4\sin^3\phi}{3} - \;.\;.\;.\right\}$$

$$= a\left\{\frac{\log\tan\dfrac{90^\circ+\phi}{2}}{M} - e^2\sin\phi - \frac{e^4\sin^3\phi}{3} - \;.\;.\;.\right\}$$

where M is the Modulus of Common Logarithms $= \cdot 4343$.

EXAMPLES FOR EXERCISES

1. If α cygni (declination $45^\circ\ 0'\ 7''$ N., right ascension 20 hr. 38 min. 50 sec.) crossed the meridian (north of zenith) of a station in the northern hemisphere at 5 hr. 10 min. 40 sec. a.m., G.M.T., on 12th January, 1925, find the longitude of the station.

You are given that G.S.T. at G.M.N. on that day was 19 hr. 25 min. 50 sec., and that sidereal time gains on mean time 10 sec. per hour.

Also find the latitude if the corrected meridian altitude was $58^\circ\ 14'\ 25''$. (London University, Cartography, 1927.)

2. A star whose right ascension is 13 hr. 24 min. 12 sec. is observed crossing the South meridian at a place in the northern hemisphere of approximate longitude 30° E. Find the local sidereal time of transit, and the *exact* longitude if G.S.T. at the moment of transit is 11 hr. 16 min. 2 sec. Find also the L.M.T. of the transit if G.S.T. at G.M.N. is 4 hr. 32 min. 45 sec., taking the difference between mean and sidereal time intervals at 10 sec. per hour. (London University, B.A.Hons., 1925.)

3. If the left limb of the sun was due south in England on 14th May, 1928, at 12 hr. 13 min. 10 sec. G.M.T., and the observed altitude of the lower limb at transit was $56^\circ\ 6'\ 5''$, find the latitude and longitude of the station given. Declination at G.M.N. $= 18^\circ\ 38'\ 11{\cdot}4''$ increasing $36''$ per hour; equation of time at G.M.N. = 3 min. 47 sec. to be subtracted from apparent time, and increasing 0·003 sec. per hour; mean time of semi-diameter passing meridian $= 1'\ 6{\cdot}9''$; semi-diameter $= 15'\ 51''$; refraction $= 38''$; parallax $= 5''$; and index error $= 1'\ 5''$ to be added.

4. At about what time will Procyon cross the meridian on 7th March?

5. At exactly what Greenwich mean time will Procyon (right ascension = 7 hr. 35 min. 32·3 sec.) cross the meridian of a station in 8° W. longitude on 7th March, if G.S.T. at G.M.N. on that day is 22 hr. 59 min. 49 sec., and sidereal time gains 9·86 sec. per hour on mean time?

6. If the observed meridian altitude of β Ursae Minoris (declination $74° 26' 59''$ N., right ascension 14 hr. 50 min. 56 sec.) at a station was $45° 0' 27''$ and the index error was $1' 5''$ to be subtracted, find all possible values of the latitude. Take refraction $= 57''$.

7. The altitudes shown in the table were observed in a "paired" observation for latitude. Find the latitude.

Star	Declination	Altitude	Refraction
A	63° 55′ 41″ S.	58° 1′ 30″ S.	− 36″
B	11° 1′ 16″ N.	47° 3′ 0″ N.	− 54″

8. The angles and distances given in the table were observed in a small closed traverse. Taking the bearing of *AB* as zero,

Pt.	Angle	Distance
	° ′	ft.
A		
		790·2
B	278 24	
		556·4
C	252 32	
		482·6
D	258 43	
		664·3
A	290 19	
B		

and *A* as the origin of co-ordinates, calculate the corrected co-ordinates of the other stations. (University of London, Cartography, 1928.)

9. On 1st November, 1928, the times of transit (by a watch keeping mean time) were registered at a certain station, as follows, for two stars, and the moon's *left* limb. Find, to the nearest tenth of a second, the right ascension of the moon's left limb at the moment of transit, giving the mean of the results from the two stars. The right ascensions of the stars

were 5 hr. 21 min. 47·3 sec. and 5 hr. 33 min. 23·2 sec. respectively.

Body observed	β Tauri	Moon	ξ Tauri
Time of transit a.m.	hr. min. sec. 2 56 50	hr. min. sec. 3 6 1	hr. min. sec. 3 8 24

10. Referring to the above, the *Nautical Almanac* states that on that date the right ascension of the moon's limb at transit at Greenwich was 5 hr. 29 min. 8·7 sec., increasing 168·98 sec. per hour of longitude. Hence, find the longitude of the station.

11. Work the same question from the following data: Sidereal time of moon's semi-diameter passing meridian = 76·30 sec.; moon's right ascension at 3 a.m. G.M.T. = 5 hr. 28 min. 26·7 sec., increasing 26·97 sec. in 10 min., and G.S.T. at G.M.N. = 14 hr. 42 min. 5·9 sec., instead of the data in Q. 10.

12. Using the above results for the longitude, find how much the watch in Q. 9 was fast or slow on local mean time.

13. The angles and distances in the table were obtained in a small closed traverse. Taking the bearing of *DE* as zero,

Pt.	Angle	Distance
	° ′	
A		
		212·4
B	113 27	
		201·7
C	233 28	
		224·1
D	46 22	
		317·2
E	118 58	
		262·2
F	120 14	
		286·8
A	87 28	
B		

and the co-ordinates of *D* as zero north and 500·0 east, calculate the corrected co-ordinates of the other points.

ANSWERS TO EXERCISES

1. 108° 32′ 3″ east, 13° 14′ 32″ N.
2. Longitude = 32° 2′ 30″ E.; L.M.T. = 8 hr. 50 min. 19·8 sec. p.m.
3. Latitude = 52° 15′ 52″ N., longitude, 3° 57′ 31·5″ W.
4. About 8.30 p.m., local time.
5. 9 hr. 6 min. 12 sec. p.m., G.M.T.
6. 60° 31′ 26″ N. or 29° 25′ 24″ N.
7. 31° 56′ 36·5″ S.
8. Angles at *C* and *D* corrected; latitudes and departures closed by Bowditch's method.

Pt.	*A*	*B*	*C*	*D*
N.	0	789·7	708·0	231·1
E.	0	0	549·3	624·2

9. 5 hr. 30 min. 59·8 sec.

10 and 11. 9° 51′ 43·5″ W.

12. G.M.T. of moon's transit at station = 3 hr. 28 min. 28·6 sec. a.m. longitude correction—39 min. 26·9 sec. Hence the watch is fast on local mean time, 16 min. 59·3 sec., or about 2½ min. on Dublin time.

13.

Pt.	*D*	*E*	*F*	*A*	*B*	*C*
N.	0	317·3	444·4	297·7	120·2	154·7
E.	500	500	270·1	23·4	139·7	338·1

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PRINTED IN GREAT BRITAIN AT THE PITMAN PRESS, BATH
B9—(1243A)

BORROWER'S NO.	ISSUE DATE	BORROWER'S NO.	ISSUE DATE